UPSC Civil Services Main 1988 - Mathematics Linear Algebra

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Question 1(a) *Show that a linear transformation of a vector space $\mathcal{V}_m$ of dimension m into a vector space $\mathcal{V}_n$ of dimension n over the same field can be represented as a matrix. If $\mathbf{T}$ is a linear transformation of $\mathcal{V}_2$ into $\mathcal{V}_4$ such that $\mathbf{T}(3,1) = (4,1,2,1)$ and $\mathbf{T}(-1,2) = (3,0,-2,1)$, then find the matrix of $\mathbf{T}$.*

Solution. Let $\mathbf{v_i}, i = 1, \ldots, m$ be a basis of $\mathcal{V}_m$ and $\mathbf{w_j}, j = 1, \ldots, n$ be a basis of $\mathcal{V}_n$. If

$$\mathbf{T}(\mathbf{v_i}) = \sum_{j=1}^{n} a_{ji}\mathbf{w_j}, \quad i = 1, \ldots, m$$

then $\mathbf{T}$ corresponds to the $n \times m$ matrix $\mathbf{A}$ whose (i,j)'th entry is a_{ij}. In fact $(\mathbf{v_1}, \ldots, \mathbf{v_m}) = (\mathbf{w_1}, \ldots, \mathbf{w_n})\mathbf{A}$.

It can be easily seen that

$$\begin{aligned} \mathbf{e_1} &= (1,0) = \frac{2}{7}(3,1) - \frac{1}{7}(-1,2) \\ \mathbf{e_2} &= (0,1) = \frac{1}{7}(3,1) + \frac{3}{7}(-1,2) \end{aligned}$$

and therefore

$$\begin{aligned}
\mathbf{T}(\mathbf{e_1}) &= \frac{2}{7}(4,1,2,1) - \frac{1}{7}(3,0,-2,1) \\
&= \frac{1}{7}(5,2,6,1) \\
&= \frac{1}{7}(5\mathbf{e_1^*} + 2\mathbf{e_2^*} + 6\mathbf{e_3^*} + \mathbf{e_4^*}) \\
\mathbf{T}(\mathbf{e_2}) &= \frac{1}{7}(4,1,2,1) + \frac{3}{7}(3,0,-2,1) \\
&= \frac{1}{7}(13,1,-4,4) \\
&= \frac{1}{7}(13\mathbf{e_1^*} + \mathbf{e_2^*} - 4\mathbf{e_3^*} + 4\mathbf{e_4^*})
\end{aligned}$$

Thus $\mathbf{T}$ corresponds to the matrix $\frac{1}{7}\begin{pmatrix} 5 & 13 \\ 2 & 1 \\ 6 & -4 \\ 7 & 4 \end{pmatrix}$ w.r.t. the standard basis. ∎

Question 1(b) *If $\mathcal{M}, \mathcal{N}$ are finite dimensional subspaces of $\mathcal{V}$, then show that* $\dim(\mathcal{M} + \mathcal{N}) = \dim \mathcal{M} + \dim \mathcal{N} - \dim(\mathcal{M} \cap \mathcal{N})$.

Solution. Let $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\}$ be a basis of $\mathcal{M} \cap \mathcal{N}$ where $\dim(\mathcal{M} \cap \mathcal{N}) = r$. Complete $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\}$ to a basis $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}, \mathbf{v_1}, \ldots, \mathbf{v_m}\}$ of $\mathcal{M}$, where $\dim \mathcal{M} = m + r$. Complete $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\}$ to a basis $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}, \mathbf{w_1}, \ldots, \mathbf{w_n}\}$ of $\mathcal{N}$, where $\dim \mathcal{N} = n + r$. We shall show that $\mathscr{B} = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}, \mathbf{v_1}, \ldots, \mathbf{v_m}, \mathbf{w_1}, \ldots, \mathbf{w_n}\}$ is a basis of $\mathcal{M} + \mathcal{N}$, proving the result.

If $\mathbf{u} \in \mathcal{M} + \mathcal{N}$, then $\mathbf{u} = \mathbf{v} + \mathbf{w}$ for some $\mathbf{v} \in \mathcal{M}, \mathbf{w} \in \mathcal{N}$. Since $\mathscr{B}$ is a superset of the bases of $\mathcal{M}, \mathcal{N}$, $\mathbf{v}, \mathbf{w}$ can be written as linear combination of elements of $\mathscr{B} \Rightarrow \mathbf{u}$ can be written as a linear combination of elements of $\mathscr{B}$. Thus $\mathscr{B}$ generates $\mathcal{M} + \mathcal{N}$.

We now show that the set $\mathscr{B}$ is linearly independent. If possible let

$$\sum_{i=1}^{n} \alpha_i \mathbf{v_i} + \sum_{i=1}^{m} \beta_i \mathbf{w_i} + \sum_{i=1}^{r} \gamma_i \mathbf{u_i} = \mathbf{0}$$

Since $\sum_{i=1}^{n} \alpha_i \mathbf{v_i} = -\sum_{i=1}^{m} \beta_i \mathbf{w_i} - \sum_{i=1}^{r} \gamma_i \mathbf{u_i}$ it follows that $\sum_{i=1}^{n} \alpha_i \mathbf{v_i} \in \mathcal{N}$. Therefore $\sum_{i=1}^{n} \alpha_i \mathbf{v_i} \in \mathcal{M} \cap \mathcal{N} \Rightarrow \sum_{i=1}^{n} \alpha_i \mathbf{v_i} = \sum_{i=1}^{r} \eta_i \mathbf{u_i}$ for $\eta_i \in \mathbb{R}$. This means that $\sum_{i=1}^{n} \alpha_i \mathbf{v_i} - \sum_{i=1}^{r} \eta_i \mathbf{u_i} = 0$. But $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}, \mathbf{v_1}, \ldots, \mathbf{v_m}\}$ are linearly independent, so $\alpha_i = 0, 1 \leq i \leq n$. Similarly we can show that $\beta_i = 0, 1 \leq i \leq m$. Then the linear independence of $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\}$ shows that $\gamma_i = 0, 1 \leq i \leq r$. Thus the vectors in $\mathscr{B}$ are linearly independent and form a basis of $\mathcal{M} + \mathcal{N}$, showing that the dimension of $\mathcal{M} + \mathcal{N}$ is $m + n + r = (m + r) + (n + r) - r$, which completes the proof. ∎

Question 1(c) *Determine a basis of the subspace spanned by the vectors* $\mathbf{v_1} = (1, 2, 3), \mathbf{v_2} = (2, 1, -1), \mathbf{v_3} = (1, -1, -4), \mathbf{v_4} = (4, 2, -2)$.

Solution. $\mathbf{v_1}, \mathbf{v_2}$ are linearly independent because if $\alpha\mathbf{v_1} + \beta\mathbf{v_2} = \mathbf{0}$ then $\alpha + 2\beta = 0, 2\alpha + \beta = 0, 3\alpha - \beta = 0 \Rightarrow \alpha = \beta = 0$. If $\mathbf{v_3} = \alpha\mathbf{v_1} + \beta\mathbf{v_2}$, then the three linear equations $\alpha + 2\beta = 1, 2\alpha + \beta = -1, 3\alpha - \beta = -4$ should be consistent — clearly $\alpha = -1, \beta = 1$ satisfy all three, showing $\mathbf{v_3} = \mathbf{v_2} - \mathbf{v_1}$. Again suppose $\mathbf{v_4} = \alpha\mathbf{v_1} + \beta\mathbf{v_2}$, then the three linear equations $\alpha + 2\beta = 4, 2\alpha + \beta = 2, 3\alpha - \beta = -2$ should be consistent — clearly $\alpha = 0, \beta = 2$ satisfy all three, showing $\mathbf{v_4} = 2\mathbf{v_2}$.

Hence $\{\mathbf{v_1}, \mathbf{v_2}\}$ is a basis for the vector space generated by $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$. ∎

Question 2(a) *Show that it is impossible for* $\mathbf{S} = \begin{pmatrix} a_1 & b \\ b & a_2 \end{pmatrix}, b \neq 0$ *to have identical eigenvalues.*

Solution. We know given $\mathbf{S}$ symmetric $\exists\mathbf{O}$ orthogonal so that $\mathbf{O}'\mathbf{S}\mathbf{O} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where λ_1, λ_2 are eigenvalues of $\mathbf{S}$. If $\lambda_1 = \lambda_2$, then we have $\mathbf{S} = \mathbf{O}'^{-1}(\lambda\mathbf{I})\mathbf{O}^{-1} = \lambda(\mathbf{O}\mathbf{O}')^{-1} = \lambda\mathbf{I} \Rightarrow \mathbf{S} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. Thus if $b \neq 0$, $\mathbf{S}$ cannot have identical eigenvalues. ∎

Question 2(b) *Prove that the eigenvalues of a Hermitian matrix are all real and the eigenvalues of a skew-Hermitian matrix are either zero or pure imaginary.*

Solution. See question 2(a), year 1998. ∎

Question 2(c) *If* $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$ *for all* $\mathbf{x} \neq \mathbf{0}$, $\mathbf{A}$ *symmetric, then for all* $\mathbf{y} \neq \mathbf{0}$ $\mathbf{y}'\mathbf{A}^{-1}\mathbf{y} > 0$. *If* λ *is the largest eigenvalue of* $\mathbf{A}$, *then*

$$\lambda = \sup_{\substack{\mathbf{x} \in \mathbb{R}^n \\ \mathbf{x} \neq 0}} \frac{\mathbf{x}'\mathbf{A}\mathbf{x}}{\mathbf{x}'\mathbf{x}}$$

.

Solution. Clearly $\mathbf{A} = \mathbf{A}'\mathbf{A}^{-1}\mathbf{A} \therefore \mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{A}'\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{y}'\mathbf{A}^{-1}\mathbf{y}$ where $\mathbf{y} = \mathbf{A}\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$. Since $|\mathbf{A}| \neq 0$, any vector $\mathbf{y}$ can be written as $\mathbf{A}\mathbf{x}$, by taking $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$. Thus $\mathbf{x}'\mathbf{A}\mathbf{x} > 0 \Rightarrow \mathbf{y}'\mathbf{A}^{-1}\mathbf{y} > 0$ for all $\mathbf{y} \neq \mathbf{0}$.

Let $M = \sup_{\substack{\mathbf{x} \in \mathbb{R}^n \\ \mathbf{x} \neq 0}} \frac{\mathbf{x}'\mathbf{A}\mathbf{x}}{\mathbf{x}'\mathbf{x}}$. Let $\mathbf{O}$ be an orthogonal matrix such that $\mathbf{O}'\mathbf{A}\mathbf{O} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$.

Let $\mathbf{0} \neq \mathbf{x} = \mathbf{O}\mathbf{y}$, then $\mathbf{x}'\mathbf{x} = \mathbf{y}'\mathbf{O}'\mathbf{O}\mathbf{y} = \mathbf{y}'\mathbf{y}$. Now $\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{y}'\mathbf{O}'\mathbf{A}\mathbf{O}\mathbf{y} = \sum_i \lambda_i y_i^2 \leq \lambda\mathbf{y}'\mathbf{y}$ where λ is the largest eigenvalue of $\mathbf{A}$. Thus $\lambda \geq \frac{\mathbf{x}'\mathbf{A}\mathbf{x}}{\mathbf{y}'\mathbf{y}} = \frac{\mathbf{x}'\mathbf{A}\mathbf{x}}{\mathbf{x}'\mathbf{x}}$, so $\lambda \geq M$. On the other hand, if $\mathbf{x} \neq \mathbf{0}$ is an eigenvector corresponding to λ, then $\mathbf{x}'\mathbf{A}\mathbf{x} = \lambda\mathbf{x}'\mathbf{x} \Rightarrow \lambda = \frac{\mathbf{x}'\mathbf{A}\mathbf{x}}{\mathbf{x}'\mathbf{x}} \leq M$. Thus $\lambda = M$ as required. ∎

Question 3(a) *By converting* **A** *to an echelon matrix, determine its rank, where*

$$\begin{pmatrix} 0 & 0 & 1 & 2 & 8 & 9 \\ 0 & 0 & 4 & 6 & 5 & 3 \\ 0 & 2 & 3 & 1 & 4 & 7 \\ 0 & 3 & 0 & 9 & 3 & 7 \\ 0 & 0 & 5 & 7 & 3 & 1 \end{pmatrix}$$

Solution. Consider

$$\mathbf{A}' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ 1 & 4 & 3 & 0 & 5 \\ 2 & 6 & 1 & 9 & 7 \\ 8 & 5 & 4 & 3 & 3 \\ 9 & 3 & 7 & 7 & 1 \end{pmatrix}$$

Interchange the first row with the third, then third with fourth, fourth with fifth and fifth with sixth to get

$$\mathbf{A}' \sim \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 0 & 2 & 3 & 0 \\ 2 & 6 & 1 & 9 & 7 \\ 8 & 5 & 4 & 3 & 3 \\ 9 & 3 & 7 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now perform $\mathbf{R_3} - 2\mathbf{R_1}, \mathbf{R_4} - 8\mathbf{R_1}, \mathbf{R_5} - 9\mathbf{R_1}$ to get

$$\mathbf{A}' \sim \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & -2 & -5 & 9 & -3 \\ 0 & -27 & -20 & 3 & -37 \\ 0 & -33 & -20 & 7 & -44 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Interchange the second and the third row, and perform $-\frac{1}{2}\mathbf{R_2}, \frac{1}{2}\mathbf{R_3}$ to get

$$\mathbf{A}' \sim \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 1 & \frac{5}{2} & -\frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & -27 & -20 & 3 & -37 \\ 0 & -33 & -20 & 7 & -44 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Perform $\mathbf{R_4} + 27\mathbf{R_2}, \mathbf{R_5} + 33\mathbf{R_2}$ to get

$$\mathbf{A}' \sim \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 1 & \frac{5}{2} & -\frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{95}{2} & \frac{237}{2} & \frac{7}{2} \\ 0 & 0 & \frac{125}{2} & -\frac{283}{2} & \frac{11}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Operation $\mathbf{R_4} - \frac{95}{2}\mathbf{R_3}, \mathbf{R_5} - \frac{125}{2}\mathbf{R_3}$ yields

$$\mathbf{A}' \sim \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 1 & \frac{5}{2} & -\frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & -\frac{759}{4} & \frac{7}{2} \\ 0 & 0 & 0 & -\frac{851}{4} & \frac{11}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now multiply $\mathbf{R_4}$ with $-\frac{4}{759}$

$$\mathbf{A}' \sim \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 1 & \frac{5}{2} & -\frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & -\frac{14}{759} \\ 0 & 0 & 0 & -\frac{851}{4} & \frac{11}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Performing $\mathbf{R_5} + \frac{851}{4}\mathbf{R_4}$ results in

$$\mathbf{A}' \sim \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 1 & \frac{5}{2} & -\frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & -\frac{14}{759} \\ 0 & 0 & 0 & 0 & \frac{11}{2} - \frac{851\times 7}{1518} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which can be converted to

$$\mathbf{A}' \sim \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 1 & \frac{5}{2} & -\frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & -\frac{14}{759} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which is an echelon matrix. Its rank is clearly 5, so the rank of $\mathbf{A} = 5$. ∎

Question 3(b) *Given* $\mathbf{AB} = \mathbf{AC}$ *does it follow that* $\mathbf{B} = \mathbf{C}$*? Can you provide a counterexample?*

Solution. It does not follow that $\mathbf{B} = \mathbf{C}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \mathbf{AB} = \mathbf{0}$$

$\mathbf{C} = \mathbf{0} \Rightarrow \mathbf{AC} = \mathbf{0}$, but $\mathbf{B} \neq \mathbf{C}$. ∎

Question 3(c) *Find a nonsingular matrix which diagonalizes* $\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{pmatrix}$ *simultaneusly. Find the diagonal form of* $\mathbf{A}$.

Solution.

$$|\mathbf{A} - \lambda\mathbf{B}| = 0 \Rightarrow \begin{vmatrix} -2\lambda & -1-\lambda & 2\lambda \\ -1-\lambda & -1-2\lambda & 1+2\lambda \\ 2\lambda & 1+2\lambda & -3\lambda \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} -2\lambda & -1-\lambda & 0 \\ -1+\lambda & 0 & 0 \\ 2\lambda & 1+2\lambda & -\lambda \end{vmatrix} = 0$$

Thus $\lambda = 0, 1, -1$. This shows that the matrices are diagonalizable simultaneously.

We now determine $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$ such that $(\mathbf{A} - \lambda\mathbf{B})\mathbf{x_i} = \mathbf{0}, i = 1, 2, 3$. For $\lambda = 0$, let $\mathbf{x_1}' = (x_1, x_2, x_3)$ be such that $(\mathbf{A} - \lambda\mathbf{B})\mathbf{x_1} = \mathbf{0}$. Thus

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $-x_2 = 0, -x_1 - x_2 + x_3 = 0, x_2 = 0$. Thus $\mathbf{x_1}' = (1, 0, 1)$.

For $\lambda = 1$, let $\mathbf{x_2}' = (x_1, x_2, x_3)$ be such that $(\mathbf{A} - \lambda\mathbf{B})\mathbf{x_2} = \mathbf{0}$. Thus

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -3 & 3 \\ 2 & 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $-2x_1 - 2x_2 + 2x_3 = 0, -2x_1 - 3x_2 + 3x_3 = 0, 2x_1 + 3x_2 - 3x_3 = 0 \Rightarrow x_2 - x_3 = 0 \Rightarrow x_1 = 0$. Thus we may take $\mathbf{x_2}' = (0, 1, 1)$.

For $\lambda = -1$, let $\mathbf{x_3}' = (x_1, x_2, x_3)$ be such that $(\mathbf{A} - \lambda\mathbf{B})\mathbf{x_3} = \mathbf{0}$. Thus

$$\begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & -1 \\ -2 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $2x_1 - 2x_3 = 0, x_2 - x_3 = 0, -2x_1 - x_2 + 3x_3 = 0 \Rightarrow x_1 = x_2 = x_3$. Thus we may take $\mathbf{x_3}' = (1,1,1)$.

Let $\mathbf{P} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ so that

$$\mathbf{P}'\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

∎