UPSC Civil Services Main 1990 - Mathematics Linear Algebra

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June 14, 2007

1 Linear Algebra

Question 1(a) *State any definition of the determinant of an $n \times n$ matrix and show that the determinant function is multiplicative i.e.* $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$ *for any two $n \times n$ matrices* $\mathbf{A}, \mathbf{B}$. *You may assume the matrices to be real.*

Solution. Let π be a permutation of $1, \ldots, n$. Define $\operatorname{sign}(\pi)$ as follows: count the number of pairs of numbers that need to be interchanged to get to π from the identity permutation. If this is even, the sign is 1, and if it is odd, the sign is -1. Now if Π is the set of all permutations of $1, \ldots, n$, define

$$\det \mathbf{A} = \sum_{\pi \in \Pi} \operatorname{sign}(\pi) \prod_i a_{i\pi(i)}$$

where a_{ij} are the elements of $\mathbf{A}$.

Note that the $\det \mathbf{A}$ is n-linear i.e. if we perform any row or column operation on $\mathbf{A}$ the determinant is unchanged. Also, if any two rows are swapped, the sign of the determinant changes. These are simple consequences of the above definition.

Consider the $2n \times 2n$ matrix

$$\mathbf{P} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & 0 & \ldots & 0 \\ -1 & 0 & \ldots & 0 & b_{11} & \ldots & b_{1n} \\ 0 & -1 & \ldots & 0 & b_{21} & \ldots & b_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & -1 & b_{n1} & \ldots & b_{nn} \end{pmatrix}$$

Then $\det \mathbf{P} = \det \mathbf{A} \det \mathbf{B}$, because if for any permutation π, $\pi(i) > n$ for $i \leq n$, then the corresponding element of the sum is 0 as $a_{i\pi(i)} = 0$. Thus $\pi(i) \leq n$ if $i \leq n$, and consequently $\pi(j) > n$ if $j > n$. So each permutation consists of a permutation of $1, \ldots, n$ and a permutation of $n+1, \ldots, 2n$, consequently we can factor the sum, to get $\det \mathbf{P} = \det \mathbf{A} \det \mathbf{B}$.

Now we perform a series of column operations to $\mathbf{P}$ — add $b_{11}\mathbf{C}_1 + \ldots + b_{n1}\mathbf{C}_n$ to $\mathbf{C}_{n+1}$, to get

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & c_{11} & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & c_{21} & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & c_{n1} & 0 & \ldots & 0 \\ -1 & 0 & \ldots & 0 & 0 & b_{12} & \ldots & b_{1n} \\ 0 & -1 & \ldots & 0 & 0 & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & -1 & 0 & b_{n2} & \ldots & b_{nn} \end{pmatrix}$$

where $\mathbf{C} = \mathbf{AB} = (c_{ij})$. Similarly add $b_{12}\mathbf{C}_1 + \ldots + b_{n2}\mathbf{C}_n$ to $\mathbf{C}_{n+2}$, …, $b_{1n}\mathbf{C}_1 + \ldots + b_{nn}\mathbf{C}_n$ to $\mathbf{C}_{2n}$ to get

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \begin{pmatrix} -1 & 0 & \ldots & 0 \\ 0 & -1 & \ldots & 0 \\ & & \ldots & \\ 0 & 0 & \ldots & -1 \end{pmatrix} & \mathbf{0} \end{pmatrix}$$

We can now verify that $\det \mathbf{P} = \det \mathbf{C}$. Any permutation π that leads to a non-zero term in the determinant sum must have $\pi(j) = j - n$ for $j > n$, thus $p_{i\pi(i)} = -1, i > n$. Also $\pi(j) > n$ for $j \leq n$, so any such π can be written as a permutation of $1, \ldots, n$ followed by a series of swaps of the i-th number with the $(n+i)$-th number, which is $n+i$. Also $\operatorname{sign}(\pi)$ is the same as the sign of the corresponding permutation π' of $1, \ldots, n$ — we first do π' by exchanges and then additionally swap the i-th element with the $(i+n)$-th element, for each $i \leq n$. Now if n is even, this involves an even number of additional swaps, and multiply by $(-1)^n$ corresponding to $p_{i\pi(i)}$ for $i > n$, otherwise we get an odd number of additional swaps, flipping the sign, but we still multiply by $(-1)^n = -1$.

Thus $\det \mathbf{P} = \det \mathbf{C} = \det \mathbf{A} \det \mathbf{B}$. ∎

Question 1(b) *Prove Laplace's formula for simulataneous expansion of the determinant by the first row and column; that given an* $(n+1) \times (n+1)$ *matrix in the block form* $\mathbf{M} = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \mathbf{D} \end{smallmatrix}\right)$, *where* α *is a scalar,* β *is a* $1 \times n$ *matrix (a row vector),* γ *is a* $n \times 1$ *matrix (a column vector), and* $\mathbf{D}$ *is an* $n \times n$ *matrix, then* $\det \mathbf{M} = \alpha \det \mathbf{D} - \beta \mathbf{D}' \gamma'$, *where* $\mathbf{D}'$ *is the matrix of cofactors of* $\mathbf{D}$ *and* $\beta \mathbf{D}' \gamma'$ *stands for the matrix product of size* 1×1.

Solution. Let $\mathbf{M} = (a_{ij}), 1 \leq i, j \leq n+1$. Thus $\alpha = a_{11}$, $\beta = (a_{12} \ \ldots \ a_{1,n+1})$,

$\gamma = \begin{pmatrix} a_{21} \\ \vdots \\ a_{n+1,1} \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} a_{22} & \dots & a_{2,n+1} \\ \vdots & & \vdots \\ a_{n+1,2} & \dots & a_{n+1,n+1} \end{pmatrix}$.

$\det \mathbf{M} = a_{11}|\mathbf{A}_{11}| - a_{12}|\mathbf{A}_{12}| + \ldots + (-1)^n a_{1,n+1}|\mathbf{A}_{1,n+1}|$ where $\mathbf{A}_{ij}$ is the minor corresponding to a_{ij} (formed by deleting the i-th row and j-th column of $\mathbf{A}$). Clearly $\mathbf{D} = \mathbf{A}_{11}$, so $\det \mathbf{M} = \alpha \det \mathbf{D} - \sum_{j=2}^{n+1}(-1)^j a_{1j} \det \mathbf{A}_{1j}$. Now

$$|\mathbf{A}_{1j}| = \begin{vmatrix} a_{21} & a_{22} & \dots & a_{2,j-1} & a_{2,j+1} & \dots & a_{2,n+1} \\ a_{31} & a_{32} & \dots & a_{3,j-1} & a_{3,j+1} & \dots & a_{3,n+1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n+1,1} & a_{n+1,2} & \dots & a_{n+1,j-1} & a_{n+1,j+1} & \dots & a_{n+1,n+1} \end{vmatrix}$$

Let $\mathbf{B}_{ij}$ be the minor of a_{ij} in $\mathbf{D}$. Expanding $|\mathbf{A}_{1j}|$ in terms of the first column, we get

$$|\mathbf{A}_{1j}| = a_{21}|\mathbf{B}_{2j}| - a_{31}|\mathbf{B}_{3j}| + \ldots + (-1)^{n+1}a_{n+1,1}|\mathbf{B}_{n+1,1}|$$

$$\begin{aligned} \det \mathbf{M} &= \alpha \det \mathbf{D} - \sum_{j=2}^{n+1}\sum_{i=2}^{n+1} a_{1j}a_{i1}|\mathbf{B}_{ij}|(-1)^i(-1)^j \\ &= \alpha \det \mathbf{D} - (a_{12}\ a_{13}\ \dots\ a_{1,n+1})(c_{ij}) \begin{pmatrix} a_{21} \\ \vdots \\ a_{n+1,1} \end{pmatrix} \\ &= \alpha \det \mathbf{D} - \beta \mathbf{D}'\gamma \end{aligned}$$

where $c_{ij} = (-1)^{i+j}|\mathbf{B}_{ij}|$, thus $\mathbf{D}' = (c_{ij})$ is the matrix of cofactors of $\mathbf{D}$. ∎

Question 1(c) *For* $\mathbf{M}$ *as in 1(b), if* $\mathbf{D}$ *is invertible, show that* $\det \mathbf{M} = \det \mathbf{D}(\alpha - \beta \mathbf{D}^{-1}\gamma)$.

Solution. If $\mathbf{D}$ is invertible, then $\mathbf{D}\mathbf{D}' = \mathbf{D}'\mathbf{D} = (\det \mathbf{D})\mathbf{I} \Rightarrow \mathbf{D}' = \mathbf{D}^{-1}\det \mathbf{D}$. So $\det \mathbf{M} = \alpha \det \mathbf{D} - \beta \mathbf{D}'\gamma = \alpha \det \mathbf{D} - \beta \mathbf{D}^{-1} \det \mathbf{D}\gamma = \det \mathbf{D}(\alpha - \beta \mathbf{D}^{-1}\gamma)$. ∎

Question 2(a) *Write the definition of the characteristic polynomial, eigenvalues and eigenvectors of a square matrix. Also say briefly something about the importance and/or applications of these notions.*

Solution. Let $\mathbf{A}$ be an $n \times n$ real or complex matrix. The polynomial $|\mathbf{x}\mathbf{I}_n - \mathbf{A}|$ is called the characteristic polynomial of $\mathbf{A}$. The roots of this polynomial are called the eigenvalues of $\mathbf{A}$. If λ is an eigenvalue of $\mathbf{A}$, then all the non-zero vectors $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ are called eigenvectors of $\mathbf{A}$ corresponding to λ.

Many problems in mathematics and other sciences require finding eigenvalues and eigenvectors of an operator.

- Eigenvalues can be used to find a very simple matrix for an operator — either diagonal or a block diagonal form. This can be used to compute powers of matrices quickly.
- If one wishes to solve a linear differential system like $\mathbf{x}' = \mathbf{A}\mathbf{x}$, or study the local properties of a nonlinear system, finding the diagonal form of the matrix can give us a decoupled form of the system, allowing us to find the solution or understand its qualitative behavior, like its stability and oscillatory behavior.
- The calculation of Google's Pagerank is essentially the computation of the principal eigenvector (corresponding to the eigenvalue with the largest absolute value) of a very large matrix (the adjacency matrix of the web graph) — this is used to find the relative importance of documents on the World Wide Web. Similar calculations are used to compute the stationary distribution of a Markov system.
- In mechanics, the eigenvectors of the inertia tensor are used to define the principal axes of a rigid body, which are important in analyzing the rotation of the rigid body.
- Eigenvalues can be used to compute low rank approximations to matrices, which help in reducing the dimensionality of various problems. This is used in statistics and operations research to explain a large number of observables in terms of a few hidden variables + noise.
- Eigenvalues can help us determine the form of a quadric or higher dimensional surface — see the relevant section in year 1999.
- In quantum mechanics, states are represented by unit vectors, while observable quantities (like position and energy) are represented by Hermitian matrices. The basic problem in any quantum system is the determination of the eigenvalues and eigenvectors of the energy matrix. The eigenvalues are the observed values of the observable quantity, and discreteness of the eigenvalues leads to the quantization of the observed values.

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Question 2(b) *Show that a Hermitian matrix possesses a set of eigenvectors which form an orthonormal basis. State briefly how or why a general $n \times n$ complex matrix may fail to possess n linearly independent eigenvectors.*

Solution. Let $\mathbf{H}$ be Hermitian, and $\lambda_1, \ldots, \lambda_n$ its eigenvalues, not necessarily distinct. Let $\mathbf{x_1}$ with norm 1 be an eigenvector corresponding to λ_1. Then there exists (from a result analogous to the result used in question 3(a), year 1995) a unitary matrix $\mathbf{U}$ such that $\mathbf{x_1}$ is its first column. Therefore

$$\mathbf{U_1}^{-1}\mathbf{H}\mathbf{U_1} = \overline{\mathbf{U_1}}'\mathbf{H}\mathbf{U_1} = \begin{pmatrix} \lambda_1 & \mathbf{L} \\ \mathbf{0} & \mathbf{H_1} \end{pmatrix}$$

where $\mathbf{H_1}$ is $(n-1)\times(n-1)$ and $\mathbf{L}$ is $(n-1)\times 1$. Since $\overline{\mathbf{U_1}}'\mathbf{H}\mathbf{U_1}$ is Hermitian, it follows that $\mathbf{L} = \mathbf{0}$. Consequently

$$\overline{\mathbf{U_1}}'\mathbf{H}\mathbf{U_1} = \begin{pmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{H_1} \end{pmatrix}$$

Now $\mathbf{H_1}$ is Hermitian with eigenvalues $\lambda_2, \ldots, \lambda_n$. Repeating the above argument, we find $\mathbf{U_2^*}$ an $(n-1)\times(n-1)$ unitary matrix such that

$$\overline{\mathbf{U_2^*}}'\mathbf{H_1}\mathbf{U_2^*} = \begin{pmatrix} \lambda_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{H_2} \end{pmatrix}$$

If $\mathbf{U_2} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U_2^*} \end{pmatrix}$ then $\mathbf{U_2}$ is unitary, and

$$\overline{\mathbf{U_2}}'\overline{\mathbf{U_1}}'\mathbf{H}\mathbf{U_1}\mathbf{U_2} = \begin{pmatrix} \lambda_1 & 0 & \mathbf{0} \\ 0 & \lambda_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H_2} \end{pmatrix}$$

Repeating this process or by induction, we can get $\mathbf{U}$ unitary such that

$$\overline{\mathbf{U}}'\mathbf{H}\mathbf{U} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

If $\mathbf{U} = [\mathbf{C_1}, \mathbf{C_2}, \ldots, \mathbf{C_n}]$, then $\mathbf{C_1}, \mathbf{C_2}, \ldots, \mathbf{C_n}$ are eigenvectors of $\mathbf{H}$ and form an orthonormal system.

A complex matrix $\mathbf{A}$ would fail to have n eigenvectors which are linearly independent if $\mathbf{A}$ is not diagonalizable i.e. we cannot find $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix. For example if $\mathbf{A} = \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$, $c \neq 0$, and $\mathbf{x_1}, \mathbf{x_2}$ are two independent eigenvectors of $\mathbf{A}$, then $\mathbf{P} = [\mathbf{x_1}\ \mathbf{x_2}]$ would lead to $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \Rightarrow \mathbf{A} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ which is false. ∎

Question 2(c) *Define the minimal polynomial and show that a complex matrix is diagonalizable (i.e. conjugate to a diagonal matrix) if and only if the minimal polynomial has no repeated root.*

Solution. Given a complex $n \times n$ complex matrix $\mathbf{A}$, if $f(x)$ is a nonzero polynomial with complex coefficients of least degree such that $f(\mathbf{A}) = \mathbf{0}$, then $f(x)$ is called the *minimal polynomial* of $\mathbf{A}$. The Cayley-Hamilton therem tells us that any $n \times n$ complex matrix $\mathbf{A}$ satisfies the degree n polynomial equation $|\mathbf{A} - x\mathbf{I}| = 0$, so the minimal polynomial exists and is of degree $\leq n$.

A complex $n \times n$ matrix can be thought of as a linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^n$. Let $\mathbf{T} : \mathcal{V} \longrightarrow \mathcal{V}$, $\dim \mathcal{V} = n$. Let the minimal polynomial of $\mathbf{T}$ be $p(x)$, having distinct roots $c_1, \ldots, c_k$, so $p(x) = \prod_{j=1}^{k}(x - c_j)$. We shall show that $\mathbf{T}$ is diagonalizable.

If $k = 1$, then the minimal polynomial is $x - c$, thus $\mathbf{T} - c\mathbf{I} = \mathbf{0}$, so $\mathbf{T} = c\mathbf{I}$ is diagonalizable. So assume $k > 1$.

Consider the polynomials $p_j = \prod_{\substack{i=1\\ i\neq j}}^{k} \frac{x-c_i}{c_j-c_i}$. Clearly $p_j(c_i) = 0$ for $i \neq j$, and $p_i(c_i) = 1$. This implies that the polynomials $p_1, \ldots, p_k$ are linearly independent, and each one is of degree $k-1 < k$. Thus these form a basis of the space of polynomials of degree $\leq k-1$. Thus given any polynomial g of degree $\leq k-1$, $g = \sum_{i=1}^{k} \alpha_i p_i$, where $\alpha_i = g(c_i)$. In particular, $1 = \sum_{i=1}^{k} p_i, x = \sum_{i=1}^{k} c_i p_i$. Thus

$$\mathbf{I} = \sum_{i=1}^{k} p_i(\mathbf{T}), \quad \mathbf{T} = \sum_{i=1}^{k} c_i p_i(\mathbf{T})$$

Moreover $p_i(\mathbf{T})p_j(\mathbf{T}) = \mathbf{0}, i \neq j$ because $p_i(x)p_j(x)$ is divisible by the minimal polynomial of $\mathbf{T}$. Also $p_j(\mathbf{T}) \neq \mathbf{0}, 1 \leq j \leq k$, because the degree of p_j is less than k, the degree of the minimal polynomial of $\mathbf{T}$.

Set $\mathcal{V}_i = p_i(\mathbf{T})\mathcal{V}$, then $\mathcal{V} = \mathbf{I}(\mathcal{V}) = \sum_{i=1}^{k} p_i(\mathbf{T})\mathcal{V} = \mathcal{V}_1 + \ldots + \mathcal{V}_k$. We shall now show that $\mathcal{V}_i = \mathcal{V}_{c_i}$, the eigenspace of $\mathbf{T}$ with respect to c_i.

$\mathbf{v} \in \mathcal{V}_i \Rightarrow \mathbf{v} = p_i(\mathbf{T})\mathbf{w}$ for some $\mathbf{w} \in \mathcal{V}$. Since $(x-c_i)p_i$ is divisible by p, $(\mathbf{T}-c_i\mathbf{I})\mathbf{v} = \mathbf{0}$, so $\mathbf{T}\mathbf{v} = c_i\mathbf{v}$ so $\mathbf{v} \in \mathcal{V}_{c_i}$. Conversely, if $\mathbf{v} \in \mathcal{V}_{c_i}$, then $\mathbf{T}\mathbf{v} = c_i\mathbf{v}$, or $(\mathbf{T}-c_i\mathbf{I})\mathbf{v} = \mathbf{0} \Rightarrow p_j(\mathbf{T})\mathbf{v} = \mathbf{0}$ for $j \neq i$. Since $\mathbf{v} = p_i(\mathbf{T})\mathbf{v} + \ldots + p_k(\mathbf{T})\mathbf{v}$, we get $\mathbf{v} = p_i(\mathbf{T})\mathbf{v} \Rightarrow \mathbf{v} \in \mathcal{V}_i$.

Thus $\mathcal{V} = \sum_{i=1}^{k} \mathcal{V}_{c_i}$ so $\mathcal{V}$ has a basis consisting of eigenvectors, so $\mathbf{T}$ is diagonalizable.

Conversely let $\mathbf{T}$ be diagonalizable, then we shall show that the minimal polynomial of $\mathbf{T}$ has distinct roots. Let

$$\mathbf{P}^{-1}\mathbf{T}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

and out of $\lambda_1, \ldots, \lambda_n$, let $\lambda_1, \ldots, \lambda_k$ be distinct. Let $g(x) = (x-\lambda_1)\ldots(x-\lambda_k)$. Then $\mathbf{v} \in \mathcal{V} \Rightarrow \mathbf{v} = \mathbf{v_1} + \ldots + \mathbf{v_k}$ where $\mathbf{v_i} \in \mathcal{V}_{\lambda_i}$, the eigenspace of λ_i. Thus $g(\mathbf{T})(\mathbf{v}) = \mathbf{0}$, so $g(\mathbf{T}) = \mathbf{0}$. Thus $g(x)$ is divisible by the minimal polynomial of $\mathbf{T}$. Since $g(x)$ has all distinct roots, it immediately follows that the minimal polynomial also has all distinct roots. ∎

Question 3(a) *Show that a* 2×2 *matrix* $\mathbf{M} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *is expressible in the form* $\mathbf{LDU}$, *where* $\mathbf{L}$ *has the form* $\left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)$, $\mathbf{D}$ *is diagonal and* $\mathbf{U}$ *has the form* $\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$ *If and only if either* $a \neq 0$ *or* $a = b = c = 0$. *Also show that when* $a \neq 0$ *the factorization* $\mathbf{M} = \mathbf{LDU}$ *is unique.*

Solution. Given $\mathbf{M} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, suppose $\mathbf{M} = \mathbf{LDU} = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1 & 0 \\ a_1\alpha & a_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1 & a_1\beta \\ a_1\alpha & a_1\alpha\beta + a_2 \end{smallmatrix}\right)$. Thus $\mathbf{M} = \mathbf{LDU} \Rightarrow a_1 = a,\ a_1\beta = b,\ a_1\alpha = c,\ a_1\alpha\beta + a_2 = d$. Thus if $a = 0$, then $b = c = 0$ and $d = a_2$, In this case, $\mathbf{M} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$ whatever α, β may be, i.e. $\mathbf{M}$ can be represented as $\mathbf{LDU}$ in infinitely many ways.

If $a \neq 0$, then $a_1 = a,\ \beta = \frac{b}{a},\ \alpha = \frac{c}{a},\ a_2 = d - \frac{bc}{a}$ are uniquely determined. Thus $\mathbf{M} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & d-\frac{bc}{a} \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{smallmatrix}\right)$ and has a unique representation.

Conversely, if $\mathbf{M} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right)$, i.e. $a = b = c = 0$, then $\mathbf{M} = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$ for any $\alpha, \beta \in \mathbb{R}$. If $\mathbf{M} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $a \neq 0$, then $\mathbf{M} = \mathbf{LDU}$ with $\mathbf{L} = \left(\begin{smallmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{smallmatrix}\right)$, $\mathbf{D} = \left(\begin{smallmatrix} a & 0 \\ 0 & d-\frac{bc}{a} \end{smallmatrix}\right)$, $\mathbf{U} = \left(\begin{smallmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{smallmatrix}\right)$ as shown above. ∎

Question 3(b) *Suppose a real matrix has eigenvalue λ, possibly complex. Show that there exists a real eigenvector for λ if and only if λ is real.*

Solution. If λ is real, then the $n \times n$ matrix $\mathbf{A} - \lambda\mathbf{I}$ defines a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Since $|\mathbf{A} - \lambda\mathbf{I}| = 0$, the rows are linearly dependent, so there exists $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$ such that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Thus there exists a real eigenvector for λ.

Conversely, suppose $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$. then $\lambda\mathbf{x} = \mathbf{A}\mathbf{x} = \overline{\mathbf{A}\mathbf{x}} = \overline{\lambda\mathbf{x}} = \overline{\lambda}\mathbf{x} \Rightarrow (\lambda - \overline{\lambda})\mathbf{x} = \mathbf{0} \Rightarrow \lambda - \overline{\lambda} = 0 \because \mathbf{x} \neq \mathbf{0} \Rightarrow \lambda = \overline{\lambda}$ i.e. λ is real. ∎

Question 3(c) *If a 2×2 matrix $\mathbf{A}$ has order n, i.e. $\mathbf{A}^n = \mathbf{I}_2$, then show that $\mathbf{A}$ is conjugate to the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ where $\theta = \frac{2\pi m}{n}$ for some integer m.*

Solution. Note: $\mathbf{A}$ has to be real, otherwise the result is false: if α_1, α_2 are two distinct n-th roots of unity such that $\alpha_1 \neq \overline{\alpha_2}$, then $\mathbf{A} = \left(\begin{smallmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{smallmatrix}\right)$ has order n, but $\mathbf{A}$ is not conjugate to $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ whose eigenvalues are complex conjugates of each other.

$\mathbf{A}^n = \mathbf{I} \Rightarrow$ eigenvalues of $\mathbf{A}$ are n-th roots of unity. If $\mathbf{A}$ has repeated eigenvalues, then these can be 1 or -1, because eigenvalues of real matrices are complex conjugates of each other, so the repeated eigenvalues must be real, and they also must be roots of 1.

<u>Case 1:</u> $\mathbf{A}$ has eigenvalues $1, 1$. There exists $\mathbf{P}$ non-singular such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$. Now $(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^n = \left(\begin{smallmatrix} 1 & nc \\ 0 & 1 \end{smallmatrix}\right) = \mathbf{P}^{-1}\mathbf{A}^n\mathbf{P} = \mathbf{P}^{-1}\mathbf{I_2}\mathbf{P} = \mathbf{I_2}$, so $nc = 0 \Rightarrow c = 0$. Thus $\mathbf{A}$ is conjugate to $\mathbf{I_2} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta = \frac{2\pi n}{n}$.

<u>Case 2:</u> $\mathbf{A}$ has eigenvalues $-1, -1$. There exists $\mathbf{P}$ non-singular such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \left(\begin{smallmatrix} -1 & c \\ 0 & -1 \end{smallmatrix}\right)$. Now $\mathbf{P}^{-1}\mathbf{A}^n\mathbf{P} = \left(\begin{smallmatrix} -1 & nc \\ 0 & -1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & nc \\ 0 & 1 \end{smallmatrix}\right)$, according as n is odd or even. But $\mathbf{A}^n = \mathbf{I_2}$, therefore n is even and $c = 0$. Thus $\mathbf{A}$ is conjugate to $-\mathbf{I_2} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta = \frac{2\pi m}{n}, m = \frac{n}{2}$.

<u>Case 3:</u> $\mathbf{A}$ has distinct eigenvalues λ_1, λ_2. Then $\lambda_1 = \overline{\lambda_2}$. If $\lambda_1 = \cos\theta + i\sin\theta$, with $\theta = \frac{2\pi m}{n}$, set $\mathbf{B} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. The eigenvalues of $\mathbf{B}$ are roots of $\begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{vmatrix} = 0 \Rightarrow \lambda = \cos\theta \pm i\sin\theta$. Since $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues λ_1, λ_2 distinct, both are conjugate to $\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ and are therefore conjugate to each other. ∎