UPSC Civil Services Main 1991 - Mathematics
Linear Algebra

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Question 1(a) *Let $\mathcal{V}(\mathbb{R})$ be the real vector space of all 2×3 matrices with real entries. Find a basis of $\mathcal{V}(\mathbb{R})$. What is the dimension of $\mathcal{V}(\mathbb{R})$.*

Solution. Let $\mathbf{A_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbf{A_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbf{A_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathbf{B_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\mathbf{B_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\mathbf{B_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Clearly $\mathbf{A_i}, \mathbf{B_i},\ i = 1, 2, 3 \in \mathcal{V}(\mathbb{R})$. These generate $\mathcal{V}(\mathbb{R})$ because

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = a_1\mathbf{A_1} + a_2\mathbf{A_2} + a_3\mathbf{A_3} + b_1\mathbf{B_1} + b_2\mathbf{B_2} + b_3\mathbf{B_3}$$

for any arbitrary element $\mathbf{A} \in \mathcal{V}(\mathbb{R})$.

They are linearly independent because if the RHS in the above equation was equal to $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $a_i = 0, b_i = 0$ for $i = 1, 2, 3$. Thus $\mathbf{A_i}, \mathbf{B_i}, i = 1, 2, 3$ is a basis for $\mathcal{V}(\mathbb{R})$ and the dimension of $\mathcal{V}(\mathbb{R})$ is 6. ∎

Question 1(b) *Let $\mathbb{C}$ be the field of complex numbers and let $\mathbf{T}$ be the function from $\mathbb{C}^3$ to $\mathbb{C}^3$ defined by*

$$\mathbf{T}(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$$

1. *Verify that $\mathbf{T}$ is a linear transformation.*
2. *If $(a, b, c) \in \mathbb{C}^3$, what are the conditions on a, b, c so that (a, b, c) is in the range of $\mathbf{T}$? What is the rank of $\mathbf{T}$?*

3. *What are the conditions on a, b, c so that (a, b, c) is in the null space of $\mathbf{T}$? What is the nullity of $\mathbf{T}$?*

Solution. $\mathbf{T}(\mathbf{e_1}) = (1, 2, -1), \mathbf{T}(\mathbf{e_2}) = (-1, 1, -2), \mathbf{T}(\mathbf{e_3}) = (2, 0, 2)$. Clearly $\mathbf{T}(\mathbf{e_1})$ and $\mathbf{T}(\mathbf{e_3})$ are linearly independent. If

$$(-1, 1, 2) = \alpha(1, 2, -1) + \beta(2, 0, 2)$$

then $\alpha+2\beta = -1, 2\alpha = 1, -\alpha+2\beta = -2$, so $\alpha = \frac{1}{2}, \beta = -\frac{3}{4}$, so $\mathbf{T}(\mathbf{e_2})$ is a linear combination of $\mathbf{T}(\mathbf{e_1})$ and $\mathbf{T}(\mathbf{e_3})$. Thus rank of $\mathbf{T}$ is 2, nullity of $\mathbf{T}$ is 1.

If (a, b, c) is in the range of $\mathbf{T}$, then $(a, b, c) = \alpha(1, 2, -1) + \beta(2, 0, 2)$. Thus $\alpha + 2\beta = a, 2\alpha = b, -\alpha + 2\beta = c$. From the first two equations, $\alpha = \frac{b}{2}, \beta = \frac{a-\frac{b}{2}}{2}$. The equations would be consistent if $-\frac{b}{2} + a - \frac{b}{2} = c$, or $a = b + c$. So the condition for (a, b, c) to belong to the range of $\mathbf{T}$ is $a = b + c$.

If $(a, b, c) \in$ null space of $\mathbf{T}$, then $a - b + 2c = 0, 2a + b = 0, -a - 2b + 2c = 0$. Thus $3a+2c = 0$, so $a = -\frac{2c}{3}, b = \frac{4c}{3}$. Thus the conditions for (a, b, c) to belong to the null space of $\mathbf{T}$ are $3a+2c = 0, 3b = 4c$. Thus the null space consists of the vectors $\{(-\frac{2c}{3}, \frac{4c}{3}, c) \mid c \in \mathbb{R}\}$, showing that the nullity of $\mathbf{T}$ is 1. ∎

Question 1(c) *If* $\mathbf{A} = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 3 \end{smallmatrix}\right)$, *express* $\mathbf{A}^6 - 4\mathbf{A}^5 + 8\mathbf{A}^4 - 12\mathbf{A}^3 + 14\mathbf{A}^2$ *as a linear polynomial in* $\mathbf{A}$.

Solution. Characteristic polynomial of $\mathbf{A}$ is $\left|\begin{smallmatrix} 1-\lambda & 2 \\ 1 & 3-\lambda \end{smallmatrix}\right| = (\lambda - 3)(\lambda - 1) - 2 = \lambda^2 - 4\lambda + 1$. By the Cayley Hamilton theorem, $\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I} = \mathbf{0}$. Dividing the given polynomial by $\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I}$, we have

$$\begin{aligned}
& \mathbf{A}^6 - 4\mathbf{A}^5 + 8\mathbf{A}^4 - 12\mathbf{A}^3 + 14\mathbf{A}^2 \\
= \; & \mathbf{A}^4(\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I}) + 7\mathbf{A}^4 - 12\mathbf{A}^3 + 14\mathbf{A}^2 \\
= \; & (\mathbf{A}^4 + 7\mathbf{A}^2)(\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I}) + 16\mathbf{A}^3 + 7\mathbf{A}^2 \\
= \; & (\mathbf{A}^4 + 7\mathbf{A}^2 + 16\mathbf{A})(\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I}) + 71\mathbf{A}^2 - 16\mathbf{A} \\
= \; & (\mathbf{A}^4 + 7\mathbf{A}^2 + 16\mathbf{A} + 71\mathbf{I})(\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I}) + 268\mathbf{A} - 71\mathbf{I}
\end{aligned}$$

Since $\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I} = \mathbf{0}$, $\mathbf{A}^6 - 4\mathbf{A}^5 + 8\mathbf{A}^4 - 12\mathbf{A}^3 + 14\mathbf{A}^2 = 268\mathbf{A} - 71\mathbf{I}$. ∎

Question 2(a) *Let* $\mathbf{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *be a linear transformation defined by* $\mathbf{T}(x_1, x_2) = (-x_2, x_1)$.

1. *What is the matrix of* $\mathbf{T}$ *in the standard basis of* $\mathbb{R}^2$*?*

2. *What is the matrix of* $\mathbf{T}$ *in the ordered basis* $\mathcal{B} = \{\alpha_\mathbf{1}, \alpha_\mathbf{2}\}$ *where* $\alpha_\mathbf{1} = (1, 2)$, $\alpha_\mathbf{2} = (1, -1)$*?*

Solution. $\mathbf{T}(\mathbf{e_1}) = (0,1) = \mathbf{e_2}$, $\mathbf{T}(\mathbf{e_2}) = (-1,0) = -\mathbf{e_1}$. Thus $(\mathbf{T}(\mathbf{e_1}), \mathbf{T}(\mathbf{e_2})) = (\mathbf{e_1}\ \mathbf{e_2})\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. So the matrix of $\mathbf{T}$ in the standard basis is $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

$\mathbf{T}(\alpha_\mathbf{1}) = (-2,1), \mathbf{T}(\alpha_\mathbf{2}) = (1,1)$. If $(a,b) = x\alpha_\mathbf{1} + y\alpha_\mathbf{2}$, then $x + y = a, 2x - y = b$, so $x = \frac{a+b}{3}, y = \frac{2a-b}{3}$. This shows that

$$\begin{array}{l} \mathbf{T}(\alpha_\mathbf{1}) = (-2,1) = -\frac{1}{3}\alpha_\mathbf{1} - \frac{5}{3}\alpha_\mathbf{2} \\ \mathbf{T}(\alpha_\mathbf{2}) = (1,1) = \frac{2}{3}\alpha_\mathbf{1} + \frac{1}{3}\alpha_\mathbf{2} \end{array}$$

Thus $(\mathbf{T}(\alpha_\mathbf{1})\ \mathbf{T}(\alpha_\mathbf{2})) = (\alpha_\mathbf{1}\ \alpha_\mathbf{2})\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{5}{3} & \frac{1}{3} \end{pmatrix}$. Consequently the matrix of $\mathbf{T}$ in the ordered basis $\mathcal{B}$ is $\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{5}{3} & \frac{1}{3} \end{pmatrix}$. ∎

Question 2(b) *Determine a non-singular matrix* $\mathbf{P}$ *such that* $\mathbf{P'AP}$ *is a diagonal matrix, where* $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$. *Is the matrix congruent to a diagonal matrix? Justify your answer.*

Solution. The quadratic form associated with $\mathbf{A}$ is $Q(x,y,z) = 2xy + 4xz + 6yz$. Let $x = X, y = X + Y, z = Z$ (thus $X = x, Y = y - x, Z = z$). Then

$$\begin{aligned} Q(X,Y,Z) &= 2X^2 + 2XY + 4XZ + 6XZ + 6YZ \\ &= 2X^2 + 2XY + 10XZ + 6YZ \\ &= 2(X + \frac{Y}{2} + \frac{5}{2}Z)^2 - \frac{Y^2}{2} - \frac{25}{2}Z^2 + YZ \\ &= 2(X + \frac{Y}{2} + \frac{5}{2}Z)^2 - \frac{1}{2}(Y - Z)^2 - 12Z^2 \end{aligned}$$

Put

$$\begin{aligned} \xi &= X + \frac{Y}{2} + \frac{5}{2}Z = \frac{x}{2} + \frac{y}{2} + \frac{5z}{2} \\ \eta &= Y - Z = -x + y - z \\ \zeta &= Z = z \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & -3 \\ 1 & \frac{1}{2} & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

$Q(x,y,z)$ transforms to $2\xi^2 - \frac{1}{2}\eta^2 - 12\zeta^2$. Thus

$$\mathbf{P'AP} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -12 \end{pmatrix}$$

with $\mathbf{P} = \begin{pmatrix} 1 & -\frac{1}{2} & -3 \\ 1 & \frac{1}{2} & -2 \\ 0 & 0 & 1 \end{pmatrix}$ Clearly $\mathbf{A}$ is congruent to a diagonal matrix as shown above. ∎

Question 2(c) *Reduce the matrix*

$$\begin{pmatrix} 1 & 3 & 4 & -5 \\ -2 & -5 & -10 & 16 \\ 5 & 9 & 33 & -68 \\ 4 & 7 & 30 & -78 \end{pmatrix}$$

to echelon form by elementary row transformations.

Solution. Let the given matrix be $\mathbf{A}$. Operations $\mathbf{R}_2 + 2\mathbf{R}_1, \mathbf{R}_3 - 5\mathbf{R}_1, \mathbf{R}_4 - 4\mathbf{R}_1 \Rightarrow$

$$\mathbf{A} \approx \begin{pmatrix} 1 & 3 & 4 & -5 \\ 0 & 1 & -2 & 6 \\ 0 & -6 & 13 & -43 \\ 0 & -5 & 14 & -58 \end{pmatrix}$$

Operations $\mathbf{R}_3 + 6\mathbf{R}_2, \mathbf{R}_4 + 5\mathbf{R}_2 \Rightarrow$

$$\mathbf{A} \approx \begin{pmatrix} 1 & 3 & 4 & -5 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 4 & -28 \end{pmatrix}$$

Operations $\mathbf{R}_4 - 4\mathbf{R}_3 \Rightarrow$

$$\mathbf{A} \approx \begin{pmatrix} 1 & 3 & 4 & -5 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Operation $\mathbf{R}_1 - 3\mathbf{R}_2 \Rightarrow$

$$\mathbf{A} \approx \begin{pmatrix} 1 & 0 & 10 & -23 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Operations $\mathbf{R}_1 - 10\mathbf{R}_3, \mathbf{R}_2 + 2\mathbf{R}_3 \Rightarrow$

$$\mathbf{A} \approx \begin{pmatrix} 1 & 0 & 0 & 47 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is the required row echelon form. The rank of $\mathbf{A}$ is 3. ∎

Question 3(a) $\mathbf{U}$ *is an n-rowed unitary matrix such that* $|\mathbf{I}-\mathbf{U}| \neq 0$, *show that the matrix* $\mathbf{H}$ *defined by* $i\mathbf{H} = (\mathbf{I}+\mathbf{U})(\mathbf{I}-\mathbf{U})^{-1}$ *is Hermitian. If* $e^{i\alpha_1}, \ldots, e^{i\alpha_n}$ *are the eigenvalues of* $\mathbf{U}$ *then* $\cot\frac{\alpha_1}{2}, \ldots, \cot\frac{\alpha_n}{2}$ *are eigenvalues of* $\mathbf{H}$.

Solution.

$$\begin{aligned}(i\mathbf{H})(\mathbf{I}-\mathbf{U}) &= (\mathbf{I}+\mathbf{U}) \\ \Rightarrow (\mathbf{I}-\overline{\mathbf{U}}')\overline{(i\mathbf{H})}' &= (\mathbf{I}+\overline{\mathbf{U}}')\end{aligned}$$

Substituting $\mathbf{I} = \overline{\mathbf{U}}'\mathbf{U}$, we have from the second equation that $\overline{\mathbf{U}}'(\mathbf{U}-\mathbf{I})\overline{(i\mathbf{H})}' = \overline{\mathbf{U}}'(\mathbf{U}+\mathbf{I})$. So $\overline{(i\mathbf{H})}' = -i\overline{\mathbf{H}}' = -(\mathbf{I}+\mathbf{U})(\mathbf{I}-\mathbf{U})^{-1} = -i\mathbf{H}$, so $\overline{\mathbf{H}}' = \mathbf{H}$, thus $\mathbf{H}$ is Hermitian.

If an eigenvalue of a nonsingular matrix $\mathbf{A}$ is λ, then λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$ $\because$ $\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \Rightarrow \lambda^{-1}\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}$, note that $\lambda \neq 0 \because |\mathbf{A}| \neq 0$. Thus the eigenvalues of $\mathbf{H}$ are

$$\begin{aligned} & \frac{1}{i}\frac{1+e^{i\alpha_j}}{1-e^{i\alpha_j}}, 1 \leq j \leq n \\ = & -i\frac{e^{i\alpha_j/2}+e^{-i\alpha_j/2}}{e^{-i\alpha_j/2}-e^{i\alpha_j/2}}, 1 \leq j \leq n \\ = & \frac{\frac{e^{i\alpha_j/2}+e^{-i\alpha_j/2}}{2}}{\frac{e^{-i\alpha_j/2}-e^{i\alpha_j/2}}{2i}}, 1 \leq j \leq n \\ = & \frac{\cot\alpha_j}{2}, 1 \leq j \leq n\end{aligned}$$

∎

Question 3(b) *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix with distinct eigenvalues* $\lambda_1, \ldots, \lambda_n$. *Show that if* $\mathbf{A}$ *is non-singular then there exist* 2^n *matrices* $\mathbf{X}$ *such that* $\mathbf{X}^2 = \mathbf{A}$. *What happens in case* $\mathbf{A}$ *is a singular matrix?*

Solution. There exists $\mathbf{P}$ non-singular such that $\mathbf{P^{-1}AP} = diagonal[\lambda_1, \ldots, \lambda_n]$.

Let $\mathbf{Y}_1 = diagonal[\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}]$, and let $\mathbf{X} = \mathbf{PYP}^{-1}$. Then $\mathbf{X}^2 = \mathbf{PYP}^{-1}\mathbf{PYP}^{-1} = \mathbf{PY^2P}^{-1} = \mathbf{A}$. Thus any of the 2^n matrices formed by choosing a sign for each of the diagonal entries from $\mathbf{X} = \mathbf{P}\ diagonal[\pm\sqrt{\lambda_1}, \ldots, \pm\sqrt{\lambda_n}]\ \mathbf{P}^{-1}$ has the same property (note that they are all distinct).

If one of the eigenvalues is zero, the number of matrices $\mathbf{X}$ would become 2^{n-1}, since we would have one less choice. ∎

Question 3(c) *Show that a real quadratic* $\mathbf{x}'\mathbf{A}\mathbf{x}$ *is positive definite if and only if there exists a non-singular matrix* $\mathbf{B}$ *such that* $\mathbf{A} = \mathbf{B}'\mathbf{B}$.

Solution. If $\mathbf{A} = \mathbf{B}'\mathbf{B}$, then $\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{B}'\mathbf{B}\mathbf{x} = \mathbf{X}'\mathbf{X}$, where $\mathbf{X} = \mathbf{B}\mathbf{x}$. Now if $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{B}\mathbf{x} \neq \mathbf{0}$, as $\mathbf{B}$ is nonsingular, and 0 is not its eigenvalue. Thus $\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{X}'\mathbf{X} > 0$, so $\mathbf{x}'\mathbf{A}\mathbf{x}$ is positive definite.

Conversely, see the result used in the solution of question 2(c), year 1992. ∎