UPSC Civil Services Main 1992 - Mathematics Linear Algebra

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Question 1(a) *Let $\mathcal{U}$ and $\mathcal{V}$ be vector spaces over a field K and let $\mathcal{V}$ be of finite dimension. Let $\mathbf{T} : \mathcal{V} \longrightarrow \mathcal{U}$ be a linear transformation, prove that* $\dim \mathcal{V} = \dim \mathbf{T}(\mathcal{V}) + \dim \text{nullity}\, \mathbf{T}$.

Solution. See question 3(a), year 1998. ∎

Question 1(b) *Let $\mathcal{S} = \{(x, y, z) \mid x + y + z = 0, x, y, z \in \mathbb{R}\}$. Prove that $\mathcal{S}$ is a subspace of $\mathbb{R}^3$. Find a basis of $\mathcal{S}$.*

Solution. $\mathcal{S} \neq \emptyset$ because $(0, 0, 0) \in \mathcal{S}$. If $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathcal{S}$ then $\alpha_1(x_1, y_1, z_1) + \alpha_2(x_2, y_2, z_2) \in \mathcal{S}$ because $(\alpha_1 x_1 + \alpha_2 x_2) + (\alpha_1 y_1 + \alpha_2 y_2) + (\alpha_1 z_1 + \alpha_2 z_2) = \alpha_1(x_1 + y_1 + z_1) + \alpha_2(x_2 + y_2 + z_2) = 0$. Thus $\mathcal{S}$ is a subspace of $\mathbb{R}^3$.

Clearly $(1, 0, -1), (1, -1, 0) \in \mathcal{S}$ and are linearly independent. Thus $\dim \mathcal{S} \geq 2$. However $(1, 1, 1) \notin \mathcal{S}$, so $\mathcal{S} \neq \mathbb{R}^3$. Thus $\dim \mathcal{S} = 2$ and $\{(1, 0, -1), (1, -1, 0)\}$ is a basis for $\mathcal{S}$. ∎

Question 1(c) *Which of the following are linear transformations?*

1. $\mathbf{T} : \mathbb{R} \longrightarrow \mathbb{R}^2$ *defined by* $\mathbf{T}(x) = (2x, -x)$.
2. $\mathbf{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ *defined by* $\mathbf{T}(x, y) = (xy, y, x)$.
3. $\mathbf{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ *defined by* $\mathbf{T}(x, y) = (x + y, y, x)$.
4. $\mathbf{T} : \mathbb{R} \longrightarrow \mathbb{R}^2$ *defined by* $\mathbf{T}(x) = (1, -1)$.

Solution.

1.

$$\begin{aligned}\mathbf{T}(\alpha x + \beta y) &= (2\alpha x + 2\beta y, -\alpha x - \beta y)\\ &= (2\alpha x, -\alpha x) + (2\beta y, -\beta y)\\ &= \alpha \mathbf{T}(x) + \beta \mathbf{T}(y)\end{aligned}$$

Thus $\mathbf{T}$ is a linear transformation.

2. $\mathbf{T}(2(1,1)) = \mathbf{T}(2,2) = (4,2,2) \neq 2\mathbf{T}(1,1) = 2(1,1,1)$ Thus $\mathbf{T}$ is not a linear transformation.

3.

$$\begin{aligned}\mathbf{T}(\alpha(x_1, y_1) + \beta(x_2 + y_2)) &= \mathbf{T}(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)\\ &= (\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2, \alpha y_1 + \beta y_2, \alpha x_1 + \beta x_2)\\ &= \alpha(x_1 + y_1, y_1, x_1) + \beta(x_2 + y_2, y_2, x_2)\\ &= \alpha \mathbf{T}(x_1, y_1) + \beta \mathbf{T}(x_2, y_2)\end{aligned}$$

Thus $\mathbf{T}$ is a linear transformation.

4. $\mathbf{T}(2(0,0)) = \mathbf{T}(0,0) = (1,-1) \neq 2\mathbf{T}(0,0)$ Thus $\mathbf{T}$ is not a linear transformation.

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Question 2(a) *Let* $\mathbf{T} : \mathcal{M}_{2,1} \longrightarrow \mathcal{M}_{2,3}$ *be a linear transformation defined by (with the usual notation)*

$$\mathbf{T}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2 & 1 & 3\\ 4 & 1 & 5\end{pmatrix}, \mathbf{T}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}6 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}$$

Find $\mathbf{T}\begin{pmatrix}x\\y\end{pmatrix}$.

Solution.

$$\begin{aligned}\begin{pmatrix}x\\y\end{pmatrix} &= x\begin{pmatrix}1\\0\end{pmatrix} - y\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}1\\1\end{pmatrix}\\ \mathbf{T}\begin{pmatrix}x\\y\end{pmatrix} &= (x-y)\begin{pmatrix}2 & 1 & 3\\ 4 & 1 & 5\end{pmatrix} + y\begin{pmatrix}6 & 1 & 0\\ 0 & 0 & 2\end{pmatrix} = \begin{pmatrix}2x+4y & x & 3x-3y\\ 4x-4y & x-y & 5x-3y\end{pmatrix}\end{aligned}$$

∎

Question 2(b) *For what values of η do the following equations*

$$\begin{aligned} x+y+z &= 1 \\ x+2y+4z &= \eta \\ x+4y+10z &= \eta^2 \end{aligned}$$

have a solution? Solve them in each case.

Solution. Since the determinant of the coefficient matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{pmatrix}$ is 0, the system has to be consistent to be solvable.

Clearly $x+4y+10z = 3(x+2y+4z)-2(x+y+z)$. Thus for the system to be consistent we must have $\eta^2 = 3\eta - 2$, or $\eta = 1, 2$.

If $\eta = 1$, then $x+y+z = 1, x+2y+4z = 1$ so $y+3z = 0$, or $y = -3z, x = 1+2z$. Thus the space of solutions is $\{(1+2z, -3z, z) \mid z \in \mathbb{R}\}$. Note that the rank of the coefficient matrix is 2, and consequently the space of solutions is one dimensional.

If $\eta = 2$, then $x+y+z = 1, x+2y+4z = 2$, so $y+3z = 1$ or $y = 1-3z$, hence $x = 2z$. Consequently, the space of solutions is $\{(2z, 1-3z, z) \mid z \in \mathbb{R}\}$. ∎

Question 2(c) *Prove that a necessary and sufficient condition of a real quadratic form* $\mathbf{x}'\mathbf{A}\mathbf{x}$ *to be positive definite is that the leading principal minors of* $\mathbf{A}$ *are all positive.*

Solution. Let all the principal minors be positive. We have to prove that the quadratic form is positive definite. We prove the result by induction.

If $n = 1$, then $a_{11}x^2 > 0 \Leftrightarrow a_{11} > 0$. Suppose as induction hypothesis the result is true for $n = m$. Let $\mathbf{S} = \begin{pmatrix} \mathbf{B} & \mathbf{B}_1 \\ \mathbf{B}_1' & k \end{pmatrix}$ be a matrix of a quadratic form in $m+1$ variables, where $\mathbf{B}$ is $m \times m$, $\mathbf{B}_1$ is $m \times 1$ and k is a single element. Since all principle minors of $\mathbf{B}$ are leading principal minors of $\mathbf{S}$, and are hence positive, the induction hypothesis gives that $\mathbf{B}$ is positive definite. This means that there exists a non-singular $m \times m$ matrix $\mathbf{P}$ such that $\mathbf{P}'\mathbf{B}\mathbf{P} = \mathbf{I}_m$ (We shall prove this presently). Let $\mathbf{C}$ be an m-rowed column to be determined soon. Then

$$\begin{pmatrix} \mathbf{P}' & \mathbf{0} \\ \mathbf{C}' & 1 \end{pmatrix} \begin{pmatrix} \mathbf{B} & \mathbf{B}_1 \\ \mathbf{B}'_1 & k \end{pmatrix} \begin{pmatrix} \mathbf{P} & \mathbf{C} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{P}'\mathbf{B}\mathbf{P} & \mathbf{P}'\mathbf{B}\mathbf{C} + \mathbf{P}'\mathbf{B}_1 \\ \mathbf{C}'\mathbf{B}'\mathbf{P} + \mathbf{B}_1'\mathbf{P} & \mathbf{C}'\mathbf{B}\mathbf{C} + \mathbf{C}'\mathbf{B}_1 + \mathbf{B}'_1\mathbf{C} + k \end{pmatrix}$$

Let $\mathbf{C}$ be so chosen that $\mathbf{B}\mathbf{C} + \mathbf{B}_1 = \mathbf{0}$, or $\mathbf{C} = -\mathbf{B}^{-1}\mathbf{B}_1$. Then

$$\begin{pmatrix} \mathbf{P}' & \mathbf{0} \\ \mathbf{C}' & 1 \end{pmatrix} \begin{pmatrix} \mathbf{B} & \mathbf{B}_1 \\ \mathbf{B}'_1 & k \end{pmatrix} \begin{pmatrix} \mathbf{P} & \mathbf{C} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{P}'\mathbf{B}\mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}'_1\mathbf{C} + k \end{pmatrix}$$

Taking determinants, we get $|\mathbf{P}'||\mathbf{S}||\mathbf{P}| = \mathbf{B}'_1\mathbf{C} + k$, because $\mathbf{P}'\mathbf{B}\mathbf{P} = \mathbf{I}_m$, and $\mathbf{B}'_1\mathbf{C} + k$ is a single element. Since $|\mathbf{S}| > 0$, it follows that $\mathbf{B}'_1\mathbf{C} + k > 0$, so let $\mathbf{B}'_1\mathbf{C} + k = \alpha^2$. Then $\mathbf{Q}'\mathbf{S}\mathbf{Q} = \mathbf{I}_{m+1}$ with $\mathbf{Q} = \begin{pmatrix} \mathbf{P} & \mathbf{C} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \alpha^{-1} \end{pmatrix}$. Thus the quadratic forms of $\mathbf{S}$ and $\mathbf{I}_{m+1}$ take the same values. Hence $\mathbf{S}$ is positive definite, so the condition is sufficient.

The condition is necessary - Since $\mathbf{x}'\mathbf{A}\mathbf{x}$ is positive definite, there is a non-singular matrix $\mathbf{P}$ such that $\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{I} \Rightarrow |\mathbf{A}||\mathbf{P}|^2 = 1 \Rightarrow |\mathbf{A}| > 0$.

Let $1 \leq r < n$. Let $x_{r+1} = \ldots = x_n = 0$, then we obtain a quadratic form in r variables which is positive definite. Clearly the determinant of this quadratic form is the $r \times r$ principal minor of $\mathbf{A}$ which shows the result.

Proof of the result used: Let $\mathbf{A}$ be positive definite, then there exists a non-singular $\mathbf{P}$ such that $\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{I}$.

We will prove this by induction. If $n = 1$, then the form corresponding to $\mathbf{A}$ is $a_{11}x^2$ and $a_{11} > 0$, so that $\mathbf{P} = (\sqrt{a_{11}})$.

Take

$$\mathbf{P_1} = \begin{pmatrix} 1 & -a_{11}^{-1}a_{12} & 0 & \ldots & 0 \\ 0 & & & & \\ \vdots & & & (n-1)\times(n-1) & \\ 0 & & & & \end{pmatrix}$$

then

$$\mathbf{P_1'AP_1} = \begin{pmatrix} a_{11} & 0 & a_{13} & \ldots & a_{1n} \\ 0 & & & & \\ a_{13} & & & & \\ \vdots & & & (n-1)\times(n-1) & \\ a_{1n} & & & & \end{pmatrix}$$

Repeating this process, we get a non-singular $\mathbf{Q}$ such that

$$\mathbf{Q}'\mathbf{A}\mathbf{Q} = \begin{pmatrix} a_{11} & 0 & \ldots & 0 \\ \vdots & & (n-1)\times(n-1) & \\ 0 & & & \end{pmatrix}$$

Given the $(n-1)\times(n-1)$ matrix on the lower right, we get by induction $\mathbf{P}^*$ s.t. $\mathbf{P}^{*\prime}((n-1)\times(n-1)$ matrix$)\mathbf{P}^*$ is diagonal. Thus $\exists \mathbf{P}, |\mathbf{P}| \neq 0, \mathbf{P}'\mathbf{A}\mathbf{P} = [\alpha_1, \ldots, \alpha_n]$ say. Take $\mathbf{R} =$ diagonal$[\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}]$, then $\mathbf{R}'\mathbf{P}'\mathbf{A}\mathbf{P}\mathbf{R} = \mathbf{I_n}$.

∎

Question 3(a) *State the Cayley-Hamilton theorem and use it to find the inverse of* $\left(\begin{smallmatrix} 2 & 1 \\ 4 & 3 \end{smallmatrix}\right)$.

Solution. Let $\mathbf{A}$ be an $n \times n$ matrix. If $|\lambda\mathbf{I} - \mathbf{A}| = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0$ is the characteristic equation of $\mathbf{A}$, then the Cayley-Hamilton theorem says that $\mathbf{A}^n + a_1\mathbf{A}^{n-1} + \ldots + a_n\mathbf{I} = \mathbf{0}$ i.e. a matrix satisfies its characteristic equation.

The characteristic equation of $\mathbf{A} = \left(\begin{smallmatrix} 2 & 1 \\ 4 & 3 \end{smallmatrix}\right)$ is

$$\begin{vmatrix} 2-\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 2 = 0$$

By the Cayley-Hamilton theorem, $\mathbf{A}^2 - 5\mathbf{A} + 2\mathbf{I} = \mathbf{0}$, so $\mathbf{A}(\mathbf{A} - 5\mathbf{I}) = -2\mathbf{I}$, thus $\mathbf{A}^{-1} = -\frac{1}{2}(\mathbf{A} - 5\mathbf{I})$. Thus

$$\mathbf{A}^{-1} = -\frac{1}{2}\left[\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right] = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{pmatrix}$$

∎

Question 3(b) *Transform the following into diagonal form*

$$x^2 + 2xy, 8x^2 - 4xy + 5y^2$$

and give the transformation employed.

Solution. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}$

$$\text{Let } 0 = |\mathbf{A} - \lambda\mathbf{B}| = \begin{vmatrix} 1 - 8\lambda & 1 + 2\lambda \\ 1 + 2\lambda & -5\lambda \end{vmatrix} = -5\lambda + 40\lambda^2 - 4\lambda^2 - 4\lambda - 1$$

Thus $36\lambda^2 - 9\lambda - 1 = 0$, so $\lambda = \frac{9 \pm \sqrt{81+144}}{72} = \frac{1}{3}, -\frac{1}{12}$.

Let (x_1, x_2) be the vector such that $(\mathbf{A} - \lambda\mathbf{B})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$ with $\lambda = \frac{1}{3}$. Thus $-\frac{5}{3}x_1 + \frac{5}{3}x_2 = 0 \Rightarrow x_1 = x_2$. We take $\mathbf{x_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ so that $(\mathbf{A} - \lambda\mathbf{B})\mathbf{x_1} = \mathbf{0}$ with $\lambda = \frac{1}{3}$. Similarly, if (x_1, x_2) is the vector such that $(\mathbf{A} - \lambda\mathbf{B})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$ with $\lambda = -\frac{1}{12}$, then $\frac{5}{3}x_1 + \frac{5}{6}x_2 = 0$, so $2x_1 + x_2 = 0$. We take $\mathbf{x_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Now

$$\begin{aligned}
\mathbf{x_1}'\mathbf{A}\mathbf{x_1} &= \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3 \\
\mathbf{x_2}'\mathbf{A}\mathbf{x_2} &= \begin{pmatrix} 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = -3 \\
\mathbf{x_1}'\mathbf{A}\mathbf{x_2} &= \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0
\end{aligned}$$

If $\mathbf{P} = (\mathbf{x_1}\ \mathbf{x_2})$, then $\mathbf{P}'\mathbf{A}\mathbf{P} = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$, thus $x^2 + 2xy \approx 3X^2 - 3Y^2$ by $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$.

Similarly

$$\begin{aligned}
\mathbf{x_1}'\mathbf{B}\mathbf{x_1} &= \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = 9 \\
\mathbf{x_2}'\mathbf{B}\mathbf{x_2} &= \begin{pmatrix} 1 & -2 \end{pmatrix}\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \end{pmatrix}\begin{pmatrix} 12 \\ -12 \end{pmatrix} = 36 \\
\mathbf{x_1}'\mathbf{B}\mathbf{x_2} &= \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 12 \\ -12 \end{pmatrix} = 0
\end{aligned}$$

Thus $\mathbf{P}'\mathbf{B}\mathbf{P} = \begin{pmatrix} 9 & 0 \\ 0 & 36 \end{pmatrix}$, so $8x^2 - 4xy + 5y^2$ is transformed to $9X^2 + 36Y^2$ by $\begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{P}\begin{pmatrix} x \\ y \end{pmatrix}$ ∎

Question 3(c) *Prove that the characteristic roots of a Hermitian matrix are all real, and the characteristic roots of a skew Hermitian matrix are all zero or pure imaginary.*

Solution. For Hermitian matrices, see question 2(c), year 1995.

If $\mathbf{H}$ is skew-Hermitian, then $i\mathbf{H}$ is Hermitian, because $\overline{(i\mathbf{H})}' = \bar{i}\overline{\mathbf{H}}' = -i\overline{\mathbf{H}}' = i\mathbf{H}$ as $\mathbf{H} = -\overline{\mathbf{H}}'$. Thus the eigenvalues of $i\mathbf{H}$ are real. Therefore the eigenvalues of $\mathbf{H}$ are $-ix$ where $x \in \mathbb{R}$. So they must be 0 (if $x = 0$) or pure imaginary. ∎