UPSC Civil Services Main 1993 - Mathematics
Linear Algebra

Sunder Lal
Retired Professor of Mathematics
Panjab University
Chandigarh

June 14, 2007

Question 1(a) *Show that the set* $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1), (0, 1, 0)\}$ *spans the vector space* $\mathbb{R}^3$ *but is not a basis set.*

Solution. The vectors $(1, 0, 0), (0, 1, 0), (1, 1, 1)$ are linearly independent, because $\alpha(1, 0, 0) + \beta(1, 1, 0) + \gamma(1, 1, 1) = \mathbf{0} \Rightarrow \alpha + \gamma = 0, \beta + \gamma = 0, \gamma = 0 \Rightarrow \alpha = \beta = \gamma = 0$.

Thus $(1, 0, 0), (1, 1, 0), (1, 1, 1)$ is a basis of $\mathbb{R}^3$, as $\dim_{\mathbb{R}} \mathbb{R}^3 = 3$.

Any set containng a basis spans the space, so S spans $\mathbb{R}^3$, but it is not a basis because the four vectors are not linearly independent, in fact $(1, 1, 0) = (1, 0, 0) + (0, 1, 0)$. ∎

Question 1(b) *Define rank and nullity of a linear transformation. If* $\mathcal{V}$ *is a finite dimensional vector space and* $\mathbf{T}$ *is a linear operator on* $\mathcal{V}$ *such that* $\operatorname{rank} \mathbf{T}^2 = \operatorname{rank} \mathbf{T}$*, then prove that the null space of* $\mathbf{T}$ *is equal to the null space of* $\mathbf{T}^2$*, and the intersection of the range space and null space of* $\mathbf{T}$ *is the zero subspace of* $\mathcal{V}$*.*

Solution. The dimension of the image space $\mathbf{T}(\mathcal{V})$ is called rank of $\mathbf{T}$. The dimension of the vector space kernel of $\mathbf{T} = \{\mathbf{v} \mid \mathbf{T}(\mathbf{v}) = \mathbf{0}\}$ is called the nullity of $\mathbf{T}$.

Now $\mathbf{v} \in$ null space of $\mathbf{T} \Rightarrow \mathbf{T}(\mathbf{v}) = \mathbf{0} \Rightarrow \mathbf{T}^2(\mathbf{v}) = \mathbf{0} \Rightarrow \mathbf{v} \in$ null space of $\mathbf{T}^2$. Thus null space of $\mathbf{T} \subseteq$ null space of $\mathbf{T}^2$. But we are given that $\operatorname{rank} \mathbf{T} = \operatorname{rank} \mathbf{T}^2$, so therefore nullity of $\mathbf{T}$ = nullity of $\mathbf{T}^2$, because of the nullity theorem — $\operatorname{rank} \mathbf{T} + \operatorname{nullity} \mathbf{T} = \dim \mathcal{V}$. Thus null space of $\mathbf{T}$ = null space of $\mathbf{T}^2$.

Finally if $\mathbf{v} \in$ range of $\mathbf{T}$, and $\mathbf{v} \in$ null space of $\mathbf{T}$, then $\mathbf{v} = \mathbf{T}(\mathbf{w})$ for some $\mathbf{w} \in \mathcal{V}$. Now

$$\begin{aligned} \mathbf{T}^2(\mathbf{w}) = \mathbf{T}(\mathbf{v}) &\Rightarrow \mathbf{w} \in \text{ null space of } \mathbf{T}^2 \\ &\Rightarrow \mathbf{w} \in \text{ null space of } \mathbf{T} \\ &\Rightarrow \mathbf{0} = \mathbf{T}(\mathbf{w}) = \mathbf{v} \end{aligned}$$

Thus range of $\mathbf{T} \cap$ null space of $\mathbf{T} = \{\mathbf{0}\}$. ∎

Question 1(c) *If the matrix of a linear operator* $\mathbf{T}$ *on* $\mathbb{R}^2$ *relative to the standard basis* $\{(1,0),(0,1)\}$ *is* $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$*, find the matrix of* $\mathbf{T}$ *relative to the basis* $\mathbf{B} = \{(1,1),(-1,1)\}$.

Solution. Let $\mathbf{v_1} = (1,1), \mathbf{v_2} = (-1,1)$. Then $\mathbf{T}(\mathbf{v_1}) = (11)\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right) = (2,2) = 2\mathbf{v_1}$. $\mathbf{T}(\mathbf{v_2}) = (-11)\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right) = (0,0) = \mathbf{0}$. So $(\mathbf{T}(\mathbf{v_1}), \mathbf{T}(\mathbf{v_2}) = (\mathbf{v_1})\ \mathbf{v_2})\left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right)$, so the matrix of $\mathbf{T}$ relative to the basis $\mathbf{B}$ is $\left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right)$. ∎

Question 2(a) *Prove that the inverse of* $\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{B} & \mathbf{C}\end{pmatrix}$ *is* $\begin{pmatrix}\mathbf{A}^{-1} & \mathbf{0}\\ -\mathbf{C}^{-1}\mathbf{B}\mathbf{A}^{-1} & \mathbf{C}^{-1}\end{pmatrix}$ *where* $\mathbf{A}, \mathbf{C}$ *are nonsingular matrices. Hence find the inverse of* $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix}$.

Solution.
$\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}\mathbf{A}^{-1} & \mathbf{0}\\ -\mathbf{C}^{-1}\mathbf{B}\mathbf{A}^{-1} & \mathbf{C}^{-1}\end{pmatrix} = \begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{B}\mathbf{A}^{-1} - \mathbf{B}\mathbf{A}^{-1} & \mathbf{I}\end{pmatrix}$ = Identity matrix.
$\begin{pmatrix}\mathbf{A}^{-1} & \mathbf{0}\\ -\mathbf{C}^{-1}\mathbf{B}\mathbf{A}^{-1} & \mathbf{C}^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{B} & \mathbf{C}\end{pmatrix} = \begin{pmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{C}^{-1}\mathbf{B} + \mathbf{C}^{-1}\mathbf{B} & \mathbf{I}\end{pmatrix}$ = Identity matrix, which shows the result.

Let $\mathbf{A} = \mathbf{C} = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ and $\mathbf{B} = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$. Then $\mathbf{A}^{-1} = \mathbf{C}^{-1} = \left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right)$ and $\mathbf{C}^{-1}\mathbf{B}\mathbf{A}^{-1} = \left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ Thus

$$\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix}^{-1} = \begin{pmatrix}1&0&0&0\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{pmatrix}$$

∎

Question 2(b) *If* $\mathbf{A}$ *is an orthogonal matrix with the property that* -1 *is not an eigenvalue, then show that* $\mathbf{A} = (\mathbf{I} - \mathbf{S})(\mathbf{I} + \mathbf{S})^{-1}$ *for some skew symmetric matrix* $\mathbf{S}$.

Solution. We want $\mathbf{S}$ skew symmetric such that $\mathbf{A}(\mathbf{I} + \mathbf{S}) = \mathbf{I} - \mathbf{S}$ i.e. $\mathbf{A} + \mathbf{A}\mathbf{S} = \mathbf{I} - \mathbf{S}$ or $\mathbf{A}\mathbf{S} + \mathbf{S} = \mathbf{I} - \mathbf{A}$ or $(\mathbf{I} + \mathbf{A})\mathbf{S} = \mathbf{I} - \mathbf{A}$. Let $\mathbf{S} = (\mathbf{I} + \mathbf{A})^{-1}(\mathbf{I} - \mathbf{A})$, note that $\mathbf{I} + \mathbf{A}$ is invertible because if $|\mathbf{I} + \mathbf{A}| = 0$, then -1 will be an eigenvalue of $\mathbf{A}$.

Note that the two factors of $\mathbf{S}$ commute, because $(\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A}) = \mathbf{I}-\mathbf{A}^2 = (\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})$, so $(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A})^{-1} = (\mathbf{I} + \mathbf{A})^{-1}(\mathbf{I} - \mathbf{A})$.

Now

$$\begin{aligned}\mathbf{S}' &= (\mathbf{I}-\mathbf{A})'((\mathbf{I}+\mathbf{A})^{-1})' \\ &= (\mathbf{I}-\mathbf{A}')(\mathbf{I}+\mathbf{A}')^{-1} \\ &= (\mathbf{A}\mathbf{A}'-\mathbf{A}')(\mathbf{A}'\mathbf{A}+\mathbf{A}')^{-1} \\ &= (\mathbf{A}-\mathbf{I})\mathbf{A}'\mathbf{A}'^{-1}(\mathbf{A}+\mathbf{I})^{-1} \\ &= -(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1} \\ &= -(\mathbf{I}+\mathbf{A})^{-1}(\mathbf{I}-\mathbf{A}) \\ &= -\mathbf{S}\end{aligned}$$

Thus $\mathbf{S}$ is skew symmetric, so $\mathbf{A} = (\mathbf{I}-\mathbf{S})(\mathbf{I}+\mathbf{S})^{-1}$ where $\mathbf{S} = (\mathbf{I}+\mathbf{A})^{-1}(\mathbf{I}-\mathbf{A})$ ∎

Question 2(c) *Show that any two eigenvectors corresponding to distinct eigenvalues of (i) Hermitian matrices (ii) unitary matrices are orthogonal.*

Solution. We first prove that the eigenvalues of a Hermitian matrix, and therefore of a symmetric matrix, are real.

Let $\mathbf{H}$ be Hermitian, and λ be one of its eigenvalues. Let $\mathbf{x} \neq \mathbf{0}$ be an eigenvector corresponding to λ. Thus $\mathbf{Hx} = \lambda\mathbf{x}$, so $\overline{\mathbf{x}}'\mathbf{Hx} = \overline{\mathbf{x}}'\lambda\mathbf{x}$. But $\overline{(\overline{\mathbf{x}}'\mathbf{Hx})}' = (\mathbf{x}'\overline{\mathbf{H}}\overline{\mathbf{x}})' = \overline{\mathbf{x}}'\overline{\mathbf{H}}'\mathbf{x} = \overline{\mathbf{x}}'\mathbf{Hx}$, because $\overline{\mathbf{H}}' = \mathbf{H}$. Note that $(\overline{\mathbf{x}}'\mathbf{Hx})' = \overline{\mathbf{x}}'\mathbf{Hx}$, since it is a single element, therefore $\overline{\mathbf{x}}'\mathbf{Hx}$ is real. Similarly $\overline{\mathbf{x}}'\mathbf{x} \neq 0$ is real, so $\lambda = \frac{\overline{\mathbf{x}}'\mathbf{Hx}}{\overline{\mathbf{x}}'\mathbf{x}}$ is real.

Let $\mathbf{H}$ be Hermitian, $\mathbf{Hx}_1 = \lambda_1\mathbf{x}_1$, $\mathbf{Hx}_2 = \lambda_2\mathbf{x}_2$ with $\lambda_1 \neq \lambda_2$. Clearly $\overline{\mathbf{x}}_2'\mathbf{Hx}_1 = \lambda_1\overline{\mathbf{x}}_2'\mathbf{x}_1$, $\overline{\mathbf{x}}_1'\mathbf{Hx}_2 = \lambda_2\overline{\mathbf{x}}_1'\mathbf{x}_2$. But $(\overline{\overline{\mathbf{x}}_2'\mathbf{Hx}_1})' = \overline{\mathbf{x}}_1'\overline{\mathbf{H}}'\mathbf{x}_2 = \lambda_1\overline{\mathbf{x}}_1'\mathbf{x}_2$. So $\lambda_2\overline{\mathbf{x}}_1'\mathbf{x}_2 = \overline{\mathbf{x}}_1'\mathbf{Hx}_2 = \overline{\mathbf{x}}_1'\overline{\mathbf{H}}'\mathbf{x}_2 = \lambda_1\overline{\mathbf{x}}_1'\mathbf{x}_2$ because $\overline{\mathbf{H}}' = \mathbf{H}$. Since $\lambda_1 \neq \lambda_2$, $\overline{\mathbf{x}}_1'\mathbf{x}_2 = 0$, so $\mathbf{x}_1, \mathbf{x}_2$ are orthogonal.

Let $\mathbf{U}$ be unitary, $\mathbf{Ux}_1 = \lambda_1\mathbf{x}_1, \mathbf{Ux}_2 = \lambda_2\mathbf{x}_2$, where λ_1, λ_2 are distinct eigenvalues of $\mathbf{U}$ with corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2$. Thus $\overline{\mathbf{x}}_2'\overline{\mathbf{U}}'\mathbf{Ux}_1 = \overline{\lambda}_2\overline{\mathbf{x}}_2'\lambda_1\mathbf{x}_1$. Since $\overline{\mathbf{U}}'\mathbf{U} = \mathbf{I}$, $\overline{\lambda}_2\overline{\mathbf{x}}_2'\lambda_1\mathbf{x}_1 = \overline{\mathbf{x}}_2'\mathbf{x}_1$, so $(1-\overline{\lambda}_2\lambda_1)(\overline{\mathbf{x}}_2'\mathbf{x}_1) = 0$. But $1-\overline{\lambda}_2\lambda_1 = \overline{\lambda}_2\lambda_2 - \overline{\lambda}_2\lambda_1 = \overline{\lambda}_2(\lambda_2-\lambda_1) \neq 0$[1]. Thus $\overline{\mathbf{x}}_2'\mathbf{x}_1 = 0$, so $\mathbf{x}_1, \mathbf{x}_2$ are orthogonal. ∎

Question 3(a) *A matrix* $\mathbf{B}$ *of order* n *is of the form* $\lambda\mathbf{A}$*, where* λ *is a scalar and* $\mathbf{A}$ *has 1 everywhere except the diagonal, which has* μ*. Find* λ, μ *so that* $\mathbf{B}$ *may be orthogonal.*

Solution. $\mathbf{A} = \begin{pmatrix} \mu & 1 & \dots & 1 \\ 1 & \mu & \dots & 1 \\ \dots & & \dots & \\ 1 & 1 & \dots & \mu \end{pmatrix}$. $\mathbf{B} = \lambda\mathbf{A}$. Thus

$$\mathbf{B}'\mathbf{B} = \begin{pmatrix} \lambda\mu & \lambda & \dots & \lambda \\ \lambda & \lambda\mu & \dots & \lambda \\ \dots & & \dots & \\ \lambda & \lambda & \dots & \lambda\mu \end{pmatrix}\begin{pmatrix} \lambda\mu & \lambda & \dots & \lambda \\ \lambda & \lambda\mu & \dots & \lambda \\ \dots & & \dots & \\ \lambda & \lambda & \dots & \lambda\mu \end{pmatrix} = \mathbf{B}\mathbf{B}' = \mathbf{B}^2$$

[1] We used here the fact that all eigenvalues of a unitary matrix have modulus 1. If $\mathbf{Ux} = \lambda\mathbf{x}$, then $\overline{\mathbf{x}}'\overline{\mathbf{U}}' = \overline{\lambda}\overline{\mathbf{x}}'$. Thus $\overline{\mathbf{x}}'\overline{\mathbf{U}}'\mathbf{Ux} = \lambda\overline{\lambda}\overline{\mathbf{x}}'\mathbf{x}$, so $\overline{\mathbf{x}}'\mathbf{x} = \lambda\overline{\lambda}\overline{\mathbf{x}}'\mathbf{x}$. Now $\overline{\mathbf{x}}'\mathbf{x} \neq 0$, so $\lambda\overline{\lambda} = 1$.

Clearly each diagonal element of $\mathbf{BB}'$ is $\lambda^2\mu^2 + (n-1)\lambda^2$, and each nondiagonal element is $2\lambda^2\mu + (n-2)\lambda^2$. Thus $\mathbf{B}$ will be orthogonal if $2\lambda^2\mu + (n-2)\lambda^2 = 0, \lambda^2\mu^2 + (n-1)\lambda^2 = 1$. Since $\lambda \neq 0$, $\mu = \frac{2-n}{2} = 1 - \frac{n}{2}$, and $\lambda^2 = \frac{1}{(1-\frac{n}{2})^2+n-1} = \frac{1}{1-n+\frac{n^2}{4}+n-1} = \frac{4}{n^2}$, thus $\lambda = \pm\frac{2}{n}$. ∎

Question 3(b) *Find the rank of the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{pmatrix}$$

by reducing it to its normal form.

Solution.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Operation $\mathbf{C_2} + \mathbf{C_1}, \mathbf{C_3} - 3\mathbf{C_1}, \mathbf{C_4} - 6\mathbf{C_1} \Rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & -6 & -10 \\ 5 & 8 & -12 & -19 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 1 & -3 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Operation $\mathbf{R_2} - \mathbf{R_1} \Rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & -10 \\ 5 & 8 & -12 & -19 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 1 & -3 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Operation $\mathbf{R_3} - 2\mathbf{R_2} \Rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & -10 \\ 5 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 1 & -3 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Interchanging $\mathbf{C_3}$ and $\mathbf{C_4}$ we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -10 & -6 \\ 5 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 1 & -6 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\mathbf{R_3} - 5\mathbf{R_1} \Rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -10 & -6 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & -2 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 1 & -6 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Operation $\frac{1}{4}\mathbf{R_2} \Rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{5}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ -3 & -2 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 1 & -6 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Operation $\mathbf{C_3} + \frac{5}{2}\mathbf{C_2}, \mathbf{C_4} + \frac{3}{2}\mathbf{C_2} \Rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ -3 & -2 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 1 & -\frac{7}{2} & -\frac{3}{2} \\ 0 & 1 & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Thus the normal form of $\mathbf{A}$ is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ so $\operatorname{rank} A = 3$. $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ -3 & -2 & 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 1 & 1 & -\frac{7}{2} & -\frac{3}{2} \\ 0 & 1 & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $\mathbf{PAQ}$ is the normal form. ∎

Question 3(c) *Determine the following form as definite, semidefinite or indefinite*

$$2x_1^2 + 2x_2^2 + 3x_3^2 - 4x_2x_3 - 4x_3x_1 + 2x_1x_2$$

Solution. Completing the squares of the given form (say $Q(x_1, x_2, x_3)$):

$$\begin{aligned} Q(x_1, x_2, x_3) &= 2(x_1 + \frac{1}{2}x_2 - x_3)^2 + \frac{3}{2}x_2^2 + x_3^2 - 2x_2x_3 \\ &= 2(x_1 + \frac{1}{2}x_2 - x_3)^2 + (x_3 - x_2)^2 + \frac{1}{2}x_2^2 \end{aligned}$$

Thus Q can be written as the sum of 3 squares with positive coefficients, so it is positive definite. ∎