UPSC Civil Services Main 1996 - Mathematics
Linear Algebra

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Question 1(a) *In* $\mathbb{R}^4$ *let* $\mathcal{W}_1$ *be the space generated by* $\{(1,1,0,-1),(2,4,6,0)\}$ *and let* $\mathcal{W}_2$ *be space generated by* $\{(-1,-2,-2,2),(4,6,4,-6),(1,3,4,-3)\}$. *Find a basis for the space* $\mathcal{W}_1+\mathcal{W}_2$.

Solution. Let $\mathbf{v_1}=(1,1,0,-1)$, $\mathbf{v_2}=(2,4,6,0)$, $\mathbf{v_3}=(-1,-2,-2,2)$, $\mathbf{v_4}=(4,6,4,-6)$, $\mathbf{v_5}=(1,3,4,-3)$. Since $\mathbf{w}\in\mathcal{W}_1+\mathcal{W}_2$ can be written as $\mathbf{w}=\mathbf{w_1}+\mathbf{w_2}$, and $\mathbf{w_1}=\alpha_1\mathbf{v_1}+\alpha_2\mathbf{v_2}$ and $\mathbf{w_2}=\alpha_3\mathbf{v_3}+\alpha_4\mathbf{v_4}+\alpha_5\mathbf{v_5}$, it follows that $\mathbf{w}$ is a linear combination of $\mathbf{v_i}\Rightarrow\mathcal{W}_1+\mathcal{W}_2$ is generated by $\{\mathbf{v_i},1\le i\le 5\}$. Thus a maximal independent subset of $\{\mathbf{v_i},1\le i\le 5\}$ will be a basis of $\mathcal{W}_1+\mathcal{W}_2$.

Clearly $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent. If possible, let $\mathbf{v_3}=\lambda_1\mathbf{v_1}+\lambda_2\mathbf{v_2}$, then the four equations

$$\begin{aligned}\lambda_1+2\lambda_2 &= -1\\ \lambda_1+4\lambda_2 &= -2\\ 0\lambda_1+6\lambda_2 &= -2\\ -\lambda_1+0\lambda_2 &= 2\end{aligned}$$

should be consistent and provide us λ_1,λ_2. Clearly the third and fourth equations give us $\lambda_1=-2,\lambda_2=-\frac{1}{3}$ which do not satisfy the first two equations. Thus $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$ are linearly independent.

If possible let $\mathbf{v_4}=\lambda_1\mathbf{v_1}+\lambda_2\mathbf{v_2}+\lambda_3\mathbf{v_3}$. Then

$$\lambda_1+2\lambda_2-\lambda_3=4 \qquad (1a)$$
$$\lambda_1+4\lambda_2-2\lambda_3=6 \qquad (1b)$$
$$0\lambda_1+6\lambda_2-2\lambda_3=4 \qquad (1c)$$
$$-\lambda_1+0\lambda_2+2\lambda_3=-6 \qquad (1d)$$

Adding (1b) and (1d) we get $4\lambda_2 = 0$, so $\lambda_2 = 0$. Solving (1a) and (1b) we get $\lambda_3 = -2, \lambda_1 = 2$. These values satisfy all the four equations, so $\mathbf{v_4} = 2\mathbf{v_1} - 2\mathbf{v_3}$.

If possible let $\mathbf{v_5} = \lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2} + \lambda_3\mathbf{v_3}$. Then

$$\lambda_1 + 2\lambda_2 - \lambda_3 = 1 \tag{2a}$$
$$\lambda_1 + 4\lambda_2 - 2\lambda_3 = 3 \tag{2b}$$
$$0\lambda_1 + 6\lambda_2 - 2\lambda_3 = 4 \tag{2c}$$
$$-\lambda_1 + 0\lambda_2 + 2\lambda_3 = -3 \tag{2d}$$

Adding (2b) and (2d) we get $4\lambda_2 = 0$, so $\lambda_2 = 0$. (2c) then gives us $\lambda_3 = -2$, and (2a) now gives $\lambda_1 = -1$, which satisfies all equations. Thus $\mathbf{v_5} = -\mathbf{v_1} - 2\mathbf{v_3}$. Hence $\{(1,1,0,-1),(2,4,6,0),(-1,-2,-2,2)\}$ is a basis of $\mathcal{W}_1 + \mathcal{W}_2$. ∎

Question 1(b) *Let $\mathcal{V}$ be a finite dimensional vector space and $\mathbf{v} \in \mathcal{V}, \mathbf{v} \neq \mathbf{0}$. Show that there exists a linear functional f on $\mathcal{V}$ such that $f(\mathbf{v}) \neq 0$.*

Solution. Complete $\mathbf{v}$ to a basis of $\mathcal{V}$, say $\{\mathbf{v_1} = \mathbf{v}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$, where $\dim \mathcal{V} = n$. Define $f(\mathbf{v_j}) = \delta_{1j}$ and $f(\sum_{j=1}^{n} a_j\mathbf{v_j}) = \sum_{j=1}^{n} a_j f(\mathbf{v_j})$.

Clearly f is a linear functional over $\mathcal{V}$, and $f(\mathbf{v}) = f(\mathbf{v_1}) = 1$. Note that $f(\mathbf{v_j}) = 0, j > 1$ and if any $\mathbf{w} \in \mathcal{V}, \mathbf{w} = \sum_i a_i\mathbf{v_i}, f(\mathbf{w}) = a_1$. ∎

Question 1(c) *Let $\mathcal{V} = \mathbb{R}^3, \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ be a basis of $\mathcal{V}$. Let $\mathbf{T} : \mathcal{V} \longrightarrow \mathcal{V}$ be such that $\mathbf{T}(\mathbf{v_i}) = \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}, 1 \leq i \leq 3$. By writing the matrix of $\mathbf{T}$ w.r.t. another basis show that the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are similar.

Solution. Clearly $\mathbf{A}$ is the matrix of $\mathbf{T}$ w.r.t. the basis $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. Note that

$$[\mathbf{T}(\mathbf{v_1}), \mathbf{T}(\mathbf{v_2}), \mathbf{T}(\mathbf{v_3})] = (\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})\mathbf{A}$$

Let

$$\begin{aligned} \mathbf{w_1} &= \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3} \\ \mathbf{w_2} &= \mathbf{v_1} - \mathbf{v_2} \\ \mathbf{w_3} &= \mathbf{v_2} - \mathbf{v_3} \\ \Rightarrow \mathbf{T}(\mathbf{w_1}) &= 3\mathbf{w_1}, \mathbf{T}(\mathbf{w_2}) = \mathbf{T}(\mathbf{w_3}) = 0 \end{aligned}$$

We now show that $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ is a basis for $\mathcal{V}$, i.e. these are linearly independent.

Let $\alpha\mathbf{w_1} + \beta\mathbf{w_2} + \gamma\mathbf{w_3} = \mathbf{0}$, then $(\alpha + \beta)\mathbf{v_1} + (\alpha - \beta + \gamma)\mathbf{v_2} + (\alpha - \gamma)\mathbf{v_3} = \mathbf{0}$. But $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent, therefore $\alpha + \beta = 0, \alpha - \beta + \gamma = 0, \alpha - \gamma = 0 \Rightarrow \alpha = \beta = \gamma = 0 \Rightarrow \mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ are linearly independent.

The matrix of $\mathbf{T}$ w.r.t. the basis $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ is clearly $\mathbf{B}$. Note that the choice of $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ is suggested by the shape of $\mathbf{B}$.

If $(\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}) = (\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})\mathbf{P}, |\mathbf{P}| \neq 0$ then $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, so $\mathbf{A}$ and $\mathbf{B}$ are similar. ∎

Question 2(a) *Let $\mathcal{V} = \mathbb{R}^3$ and $\mathbf{T} : \mathcal{V} \longrightarrow \mathcal{V}$ be a linear map defined by*

$$\mathbf{T}(x, y, z) = (x + z, -2x + y, -x + 2y + z)$$

What is the matrix of $\mathbf{T}$ w.r.t. the basis $(1, 0, 1), (-1, 1, 1), (0, 1, 1)$? Using this matrix write down the matrix of $\mathbf{T}$ with respect to the basis $(0, 1, 2), (-1, 1, 1), (0, 1, 1)$.

Solution. Let $\mathbf{v_1} = (1, 0, 1), \mathbf{v_2} = (-1, 1, 1), \mathbf{v_3} = (0, 1, 1)$. $\mathbf{T}(x, y, z) = (x+z, -2x+y, -x+2y+z) = \alpha\mathbf{v_1}+\beta\mathbf{v_2}+\gamma\mathbf{v_3}$, say. This means $\alpha-\beta = x+z, \beta+\gamma = -2x+y, \alpha+\beta+\gamma = -x+2y+z$. This implies $\alpha = x+y+z, \beta = y, \gamma = -2x$. Thus $\mathbf{T}(x, y, z) = (x+y+z)\mathbf{v_1} + y\mathbf{v_2} - 2x\mathbf{v_3}$. Hence

$$[\mathbf{T}(\mathbf{v_1})\ \mathbf{T}(\mathbf{v_2})\ \mathbf{T}(\mathbf{v_3})] = [\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}] \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ -2 & 2 & 0 \end{pmatrix}$$

Let $\mathbf{w_1} = (0, 1, 2), \mathbf{w_2} = (-1, 1, 1), \mathbf{w_3} = (0, 1, 1)$. Then

$$[\mathbf{w_1}\ \mathbf{w_2}\ \mathbf{w_3}] = [\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}] \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence

$$\begin{aligned} [\mathbf{T}(\mathbf{w_1})\ \mathbf{T}(\mathbf{w_2})\ \mathbf{T}(\mathbf{w_3})] &= [\mathbf{T}(\mathbf{v_1})\ \mathbf{T}(\mathbf{v_2})\ \mathbf{T}(\mathbf{v_3})]\mathbf{P} \\ &= [\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}]\mathbf{A}\mathbf{P} \\ &= [\mathbf{w_1}\ \mathbf{w_2}\ \mathbf{w_3}]\mathbf{P}^{-1}\mathbf{A}\mathbf{P} \end{aligned}$$

where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ -2 & 2 & 0 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus the matrix of $\mathbf{T}$ w.r.t. basis $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ is

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ -2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ -2 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix}$$

∎

Question 2(b) *Let $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional vector spaces such that $\dim \mathcal{V} \geq \dim \mathcal{W}$. Show that there is always a linear map of $\mathcal{V}$ onto $\mathcal{W}$.*

Solution. Let $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_m}$ be a basis of $\mathcal{W}$, and $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ be a basis of $\mathcal{V}, n \geq m$. Define

$$\begin{aligned} \mathbf{T}(\mathbf{v_i}) &= \mathbf{w_i}, \quad i = 1, 2, \ldots, m \\ \mathbf{T}(\mathbf{v_i}) &= \mathbf{0}, \quad i = m+1, \ldots, n \end{aligned}$$

and for any $\mathbf{v} \in \mathcal{V}, \mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v_i}, \mathbf{T}(\mathbf{v}) = \sum_{i=1}^{m} \alpha_i \mathbf{T}(\mathbf{v_i})$.

Clearly $\mathbf{T} : \mathcal{V} \longrightarrow \mathcal{W}$ is linear. $\mathbf{T}$ is onto, since if $\mathbf{w} \in \mathcal{W}, \mathbf{w} = \sum_{i=1}^{m} a_i \mathbf{w_i}$, then $\mathbf{T}(\sum_{i=1}^{m} a_i \mathbf{v_i}) = \sum_{i=1}^{m} a_i \mathbf{T}(\mathbf{v_i}) = \mathbf{w}$, proving the result. ∎

Question 2(c) *Solve by Cramer's rule*

$$\begin{aligned} x + y - 2z &= 1 \\ 2x - 7z &= 3 \\ x + y - z &= 5 \end{aligned}$$

Solution.

$$D = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 0 & -7 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & -1 & -2 \\ -5 & -7 & -7 \\ 0 & 0 & -1 \end{vmatrix} = -2$$

$$x = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 3 & 0 & -7 \\ 5 & 1 & -1 \end{vmatrix}}{D} = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 3 & 0 & -7 \\ 4 & 0 & 1 \end{vmatrix}}{D} = \frac{-31}{-2} = \frac{31}{2}$$

$$y = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 2 & 3 & -7 \\ 1 & 5 & -1 \end{vmatrix}}{D} = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -3 \\ 1 & 4 & 1 \end{vmatrix}}{D} = \frac{13}{-2} = -\frac{13}{2}$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & 5 \end{vmatrix}}{D} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 0 & 0 & 4 \end{vmatrix}}{D} = \frac{-8}{-2} = 4$$

∎

Question 3(a) *Find the inverse of the matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

by computing its characteristic polynomial.

Solution. The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned} |\mathbf{A} - \lambda\mathbf{I}| &= \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} \\ &= -\lambda[-\lambda^3] - 1[1] = \lambda^4 - 1 = 0 \end{aligned}$$

Thus by the Cayley-Hamilton theorem, $\mathbf{A}^4 = \mathbf{I}$, so $\mathbf{A}^{-1} = \mathbf{A}^3$.

$$\mathbf{A}^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \mathbf{A}^{-1}$$

∎

Question 3(b) *If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices such that $\mathbf{AB} = \mathbf{BA}$, show that $\mathbf{AB}$ and $\mathbf{BA}$ have a common characteristic vector.*

Solution. Let λ be any eigenvalue of $\mathbf{A}$ and let $\mathcal{V}_\lambda$ be the eigenspace of $\mathbf{A}$ corresponding to λ. We show that $\mathbf{B}(\mathcal{V}_\lambda) \subseteq \mathcal{V}_\lambda$. Let $\mathbf{v} \in \mathcal{V}_\lambda$, then $\mathbf{A}(\mathbf{Bv}) = \mathbf{B}(\mathbf{Av}) = \mathbf{B}(\lambda\mathbf{v}) = \lambda\mathbf{Bv} \Rightarrow \mathbf{Bv} \in \mathcal{V}_\lambda$.

Consider $\mathbf{B}^* : \mathcal{V}_\lambda \longrightarrow \mathcal{V}_\lambda$ such that $\mathbf{B}^*(\mathbf{v}) = \mathbf{B}(\mathbf{v})$ — note that $\mathbf{B}^*$ is a restriction of $\mathbf{B}$ to $\mathcal{V}_\lambda$ and we have already shown that $\mathbf{B}(\mathcal{V}_\lambda) \subseteq \mathcal{V}_\lambda$.

Let μ be an eigenvalue of $\mathbf{B}^*$, then μ is also an eigenvalue of $\mathbf{B}$ (because a basis of $\mathcal{V}_\lambda$ can be extended to a basis of $\mathcal{V}$, and in this basis $\mathbf{B} = \left(\begin{smallmatrix} \mathbf{B}^* & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{smallmatrix}\right)$ for some matrices $\mathbf{C}, \mathbf{D}$). Let $\mathbf{v} \in \mathcal{V}_\lambda$ be an eigenvector of $\mathbf{B}^*$ corresponding to μ, by definition $\mathbf{v} \neq \mathbf{0}$. Then $\mathbf{Bv} = \mathbf{B}^*\mathbf{v} = \mu\mathbf{v}$. Thus $\mathbf{A}$ and $\mathbf{B}$ have a common eigenvector $\mathbf{v}$, note that $\mathbf{Av} = \lambda\mathbf{v}$ as $\mathbf{v} \in \mathcal{V}_\lambda$. ∎

Question 3(c) *Reduce to canonical form the orthogonal matrix*

$$\mathbf{O} = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

Solution. Before solving this particular problem, we present a general discussion about orthogonal matrices. An orthogonal matrix satisfies $\mathbf{O}'\mathbf{O} = \mathbf{I}$, so its determinant is 1 or -1, here we focus on the case where $|\mathbf{O}| = 1$. If λ is an eigenvalue of $\mathbf{O}$ and $\mathbf{x}$ a corresponding eigenvector, then $|\lambda|^2\mathbf{x}'\mathbf{x} = (\mathbf{Ox})'\mathbf{Ox} = \mathbf{x}'\mathbf{O}'\mathbf{Ox} = \mathbf{x}'\mathbf{x}$, so $|\lambda| = 1$. Since the characteristic polynomial has real coefficients, the eigenvalues must be real or in complex conjugate pairs. Thus for a matrix of order 3, at least one eigenvalue is real, and must be 1 or -1. Since $|\mathbf{O}| = 1$, one real value must be 1, and the three possibilities are $\{1,1,1\},\{1,-1,-1\}$ and $\{1, e^{i\theta}, e^{-i\theta}\}$.

Here we consider the third case, as the given matrix has 1 and $\frac{1}{3} \pm i\frac{2\sqrt{2}}{3}$ as eigenvalues, proved later.

Let $\mathbf{Z} = \mathbf{X_1} + i\mathbf{X_2}$ be an eigenvector corresponding to the eigenvalue $e^{i\theta}$. Let $\mathbf{X_3}$ be the eigenvector corresponding to the eigenvalue 1. Since $\mathbf{Z}$ and $\mathbf{X_3}$ correspond to different eigenvalues, these are orthogonal, i.e. $\mathbf{Z}'\mathbf{X_3} = (\mathbf{X_1'} + i\mathbf{X_2'})\mathbf{X_3} = 0 \Rightarrow \mathbf{X_1'X_3} = 0, \mathbf{X_2'X_3} = 0$. Note that $\mathbf{X_1}, \mathbf{X_2}, \mathbf{X_3}$ are real vectors. Since $\mathbf{OZ} = e^{i\theta}\mathbf{Z} = (\cos\theta + i\sin\theta)(\mathbf{X_1} + i\mathbf{X_2})$. Equating real and imaginary parts we get

$$\begin{aligned}
\mathbf{OX_1} &= \mathbf{X_1}\cos\theta - \mathbf{X_2}\sin\theta \\
\mathbf{OX_2} &= \mathbf{X_1}\sin\theta + \mathbf{X_2}\cos\theta \\
\therefore \mathbf{X_1'O'OX_1} &= (\mathbf{X_1'}\cos\theta - \mathbf{X_2'}\sin\theta)(\mathbf{X_1}\cos\theta - \mathbf{X_2}\sin\theta) \\
\Rightarrow \quad \mathbf{X_1'X_1} &= \mathbf{X_1'X_1}\cos^2\theta - \mathbf{X_2'X_1}\cos\theta\sin\theta - \mathbf{X_1'X_2}\sin\theta\cos\theta + \mathbf{X_2'X_2}\sin^2\theta \\
\Rightarrow \quad 0 &= \mathbf{X_1'X_1}\sin^2\theta - \mathbf{X_2'X_2}\sin^2\theta + 2\mathbf{X_1'X_2}\cos\theta\sin\theta \\
\Rightarrow \quad 0 &= \mathbf{X_1'X_1}\sin\theta - \mathbf{X_2'X_2}\sin\theta + 2\mathbf{X_1'X_2}\cos\theta \quad (1)
\end{aligned}$$

(Note that $\sin\theta \neq 0$ since we are considering the case where $e^{i\theta}$ is complex.) Similarly

$$\begin{aligned}
\mathbf{X_2'O'OX_1} &= (\mathbf{X_1'}\sin\theta + \mathbf{X_2'}\cos\theta)(\mathbf{X_1}\cos\theta - \mathbf{X_2}\sin\theta) \\
\Rightarrow \quad \mathbf{X_2'X_1} &= \mathbf{X_1'X_1}\sin\theta\cos\theta - \mathbf{X_1'X_2}\sin^2\theta - \mathbf{X_2'X_2}\sin\theta\cos\theta + \mathbf{X_2'X_1}\cos^2\theta \\
\Rightarrow \quad 0 &= \mathbf{X_1'X_1}\cos\theta - \mathbf{X_1'X_2}\cos\theta - 2\mathbf{X_1'X_2}\sin\theta \quad (2)
\end{aligned}$$

Multiplying (1) by $\sin\theta$ and (2) by $\cos\theta$ and adding, we get $\mathbf{X_1'X_1} - \mathbf{X_2'X_2} = 0$ or $\mathbf{X_1'X_1} = \mathbf{X_2'X_2}$, so from (2), $\mathbf{X_1X_2} = 0$, i.e. $\mathbf{X_1}, \mathbf{X_2}$ are orthogonal.

Thus $\mathbf{X_1}, \mathbf{X_2}, \mathbf{X_3}$ are mutually orthogonal. We can assume that $\mathbf{X_1'X_1} = \mathbf{X_2'X_2} = 1$, replacing $\mathbf{Z}$ by $\lambda\mathbf{Z}, \lambda \in \mathbb{R}$ if necessary. Similarly we can take $\mathbf{X_3'X_3} = 1$. Let $\mathbf{P} = [\mathbf{X_1}\ \mathbf{X_2}\ \mathbf{X_3}]$ so that $\mathbf{P'P} = \mathbf{I}$. Now

$$\begin{aligned}
\mathbf{O}[\mathbf{X_1}\ \mathbf{X_2}\ \mathbf{X_3}] &= [\mathbf{X_1}\cos\theta - \mathbf{X_2}\sin\theta, \mathbf{X_1}\sin\theta + \mathbf{X_2}\cos\theta, \mathbf{X_3}] \\
&= [\mathbf{X_1}\ \mathbf{X_2}\ \mathbf{X_3}] \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\Rightarrow \mathbf{P}^{-1}\mathbf{OP} = \mathbf{P'OP} &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

which is the canonical form of $\mathbf{O}$ when the eigenvalues are $1, e^{i\theta}, e^{-i\theta}$.

Solution of given problem.

$$\begin{aligned}
\mathbf{O} &= \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \\
|\mathbf{O}-\lambda\mathbf{I}| &= \begin{vmatrix} \frac{2}{3}-\lambda & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3}-\lambda & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3}-\lambda \end{vmatrix} = \frac{1}{27}\begin{vmatrix} 2-3\lambda & -2 & 1 \\ 2 & 1-3\lambda & -2 \\ 1 & 2 & 2-3\lambda \end{vmatrix} \\
&= \frac{1}{27}[(2-3\lambda)^2(1-3\lambda)+4(2-3\lambda)+1(3+3\lambda)+2(6-6\lambda)] \\
&= -\frac{1}{27}[27\lambda^3-45\lambda^2+45\lambda-27] \\
&= -\frac{1}{3}[(\lambda-1)(3\lambda^2-2\lambda-3)]
\end{aligned}$$

Thus $\lambda = 1,\ \frac{1}{3} \pm i\frac{2\sqrt{2}}{3}$ are eigenvalues of $\mathbf{O}$.

Thus the canonical form of $\mathbf{O}$ is derived from above, where $\cos\theta = \frac{1}{3}, \sin\theta = \frac{2\sqrt{2}}{3}$:

$$\begin{pmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix $\mathbf{P}$ can be determined as follows (this is not needed for this problem, but is given for completeness):

1. Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 1$, then

$$\begin{aligned}
-\frac{1}{3}x_1 - \frac{2}{3}x_2 + \frac{1}{3}x_3 &= 0 \\
\frac{2}{3}x_1 - \frac{2}{3}x_2 - \frac{2}{3}x_3 &= 0 \\
\frac{1}{3}x_1 + \frac{2}{3}x_2 - \frac{1}{3}x_3 &= 0
\end{aligned}$$

   Thus $x_2 = 0, x_1 - x_3 = 0$, so we can take $(1, 0, 1)$ as an eigenvector.

2. The vectors $\mathbf{X_1}, \mathbf{X_2}$ in the above discussion are determined by the requirements

$$\begin{aligned}
\mathbf{OX_1} &= \mathbf{X_1}\cos\theta - \mathbf{X_2}\sin\theta \\
\mathbf{OX_2} &= \mathbf{X_1}\sin\theta + \mathbf{X_2}\cos\theta
\end{aligned}$$

where $\cos\theta = \frac{1}{3}, \sin\theta = \frac{2\sqrt{2}}{3}$. This gives us the following equations

$$\begin{aligned}
2x_{11} - 2x_{12} + x_{13} &= x_{11} - x_{21}2\sqrt{2} && (3)\\
2x_{11} + x_{12} - 2x_{13} &= x_{12} - x_{22}2\sqrt{2} && (4)\\
x_{11} + 2x_{12} + 2x_{13} &= x_{13} - x_{23}2\sqrt{2} && (5)\\
2x_{21} - 2x_{22} + x_{23} &= x_{11}2\sqrt{2} + x_{21} && (6)\\
2x_{21} + x_{22} - 2x_{23} &= x_{12}2\sqrt{2} + x_{22} && (7)\\
x_{21} + 2x_{22} + 2x_{23} &= x_{13}2\sqrt{2} + x_{23} && (8)
\end{aligned}$$

Adding the last 3 equations, we get $\sqrt{2}x_{21} = x_{11} + x_{12} + x_{13}$. Subtracting equation (6) from (8), $\sqrt{2}x_{22} = x_{13} - x_{11}$, and from (7) $\sqrt{2}x_{23} = x_{11} - x_{12} + x_{13}$. Substituting these in the first 3 equations and simplifying, we get $x_{11} = -x_{13}$. Setting $x_{11} = 0, x_{12} = 1$, we get $(0, 1, 0), (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$ as a possible solution for $\mathbf{X_1}, \mathbf{X_2}$.

Putting these together we get

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 1 \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 1 \end{pmatrix}$$

We can now verify that $\mathbf{OP} = \mathbf{P}\begin{pmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ∎