UPSC Civil Services Main 1997 - Mathematics
Linear Algebra

Sunder Lal
Retired Professor of Mathematics
Panjab University
Chandigarh

June 14, 2007

1 Linear Algebra

Question 1(a) *Let $\mathcal{V}$ be the vector space of polynomials over $\mathbb{R}$. Find a basis and the dimension of $\mathcal{W} \subseteq \mathcal{V}$ spanned by*

$$\begin{aligned} v_1 &= t^3 - 2t^2 + 4t + 1 \\ v_2 &= 2t^3 - 3t^2 + 9t - 1 \\ v_3 &= t^3 + 6t - 5 \\ v_4 &= 2t^3 - 5t^2 + 7t + 5 \end{aligned}$$

Solution. v_1 and v_2 are linearly independent, because if $\alpha v_1 + \beta v_2 = 0$, then $\alpha + 2\beta = 0,\ -2\alpha - 3\beta = 0,\ 4\alpha + 9\beta = 0,\ \alpha - \beta = 0 \Rightarrow \alpha = \beta = 0$.

v_3 depends linearly on v_1, v_2 — if $\alpha v_1 + \beta v_2 = v_3$, then $\alpha + 2\beta = 1,\ -2\alpha - 3\beta = 0,\ 4\alpha + 9\beta = 6,\ \alpha - \beta = -5 \Rightarrow \alpha = -3, \beta = 2$ which satisfy all the equations. Thus $v_3 = -3v_1 + 2v_2$.

v_4 depends linearly on v_1, v_2 — if $\alpha v_1 + \beta v_2 = v_4$, then $\alpha + 2\beta = 2,\ -2\alpha - 3\beta = -5,\ 4\alpha + 9\beta = 7,\ \alpha - \beta = 5 \Rightarrow \alpha = 4, \beta = -1$ which satisfy all the equations. Thus $v_4 = 4v_1 - v_2$.

Thus $\dim_{\mathbb{R}} \mathcal{W} = 2$ and v_1, v_2 is a basis of $\mathcal{W}$. ∎

Question 1(b) *Verify that $\mathbf{T}(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_2)$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$. Find its range, rank, null space and nullity.*

Solution. Let $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2)$. Then

$$\begin{aligned}\mathbf{T}(\alpha\mathbf{x} + \beta\mathbf{y}) &= \mathbf{T}(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) \\ &= (\alpha x_1 + \beta y_1 + \alpha x_2 + \beta y_2, \alpha x_1 + \beta y_1 - \alpha x_2 - \beta y_2, \alpha x_2 + \beta y_2) \\ &= (\alpha(x_1 + x_2), \alpha(x_1 - x_2), \alpha x_2) + (\beta(y_1 + y_2), \beta(y_1 - y_2), \beta y_2) \\ &= \alpha\mathbf{T}(x_1, x_2) + \beta\mathbf{T}(y_1, y_2)\end{aligned}$$

Thus $\mathbf{T}$ is linear.

$$\mathbf{T}(\mathbf{e_1}) = \mathbf{T}(1,0) = (1,1,0)$$
$$\mathbf{T}(\mathbf{e_2}) = \mathbf{T}(0,1) = (1,-1,1)$$

Clearly $\mathbf{T}(\mathbf{e_1}), \mathbf{T}(\mathbf{e_2})$ are linearly independent. Since $T(\mathbb{R}^2)$ is generated by $\mathbf{T}(\mathbf{e_1})$ and $\mathbf{T}(\mathbf{e_2})$, the rank of $\mathbf{T}$ is 2.

The range of $\mathbf{T}$ $= \{\alpha\mathbf{T}(\mathbf{e_1}) + \beta\mathbf{T}(\mathbf{e_2}), \alpha, \beta \in \mathbb{R}\}$
$= \{\alpha(1,1,0) + \beta(1,-1,1)\}$
$= \{(\alpha + \beta, \alpha - \beta, \beta) \mid \alpha, \beta \in \mathbb{R}\}$

To find the null space of $\mathbf{T}$, if $\mathbf{T}(x_1, x_2) = (0,0,0)$, then $x_1 + x_2 = 0,\ x_1 - x_2 = 0,\ x_2 = 0$, so $x_1 = x_2 = 0$. Thus the null space of $\mathbf{T}$ is $\{\mathbf{0}\}$, and nullity $\mathbf{T} = 0$. ∎

Question 1(c) *Let $\mathcal{V}$ be the space of 2×2 matrices over $\mathbb{R}$. Determine whether the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{V}$ are dependent where*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & -5 \\ -4 & 0 \end{pmatrix}$$

Solution. If $\alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C} = 0$, then

$$\alpha + 3\beta + \gamma = 0 \tag{1}$$
$$2\alpha - \beta - 5\gamma = 0 \tag{2}$$
$$3\alpha + 2\beta - 4\gamma = 0 \tag{3}$$
$$\alpha + 2\beta = 0 \tag{4}$$

From (4), we get $\alpha = -2\beta$. This, together with (3) gives $\gamma = -\beta$. These satisfy (1) and (2) also, so taking $\beta = 1, \alpha = -2, \gamma = -1$ gives us $-2\mathbf{A} + \mathbf{B} - \mathbf{C} = 0$. Thus $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are dependent. ∎

Question 2(a) *Let $\mathbf{A}$ be an $n \times n$ matrix such that each diagonal entry is μ and each off-diagonal entry is 1. If $\mathbf{B} = \lambda\mathbf{A}$ is orthogonal, determine λ, μ.*

Solution. Clearly $\mathbf{A}$ is symmetric. Let $\mathbf{A} = (a_{ij})$. $\mathbf{B}'\mathbf{B} = \mathbf{B}\mathbf{B}' = \lambda^2\mathbf{A}^2 = \mathbf{I} \implies \sum_{k=1}^{n} \lambda^2 a_{ik} a_{kj} = \delta_{ij}$

Taking $i = j = 1$, we get $\lambda^2(\mu^2 + n - 1) = 1$ Taking $i = 1, j = 2$, we get $\lambda^2(2\mu + n - 2) = 0$. Thus $\mu = -(n-2)/2$ and $\lambda^2[(n-2)^2/4 + n - 1] = 1$. Simplifying, $\lambda^2[n^2 - 4n + 4 + 4n - 4]/4 = 1$, which means $\lambda^2 = \frac{4}{n^2}$, or $\lambda = \pm\frac{2}{n}$. ∎

Question 2(b) *Show that*

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 3 \end{pmatrix}$$

is diagonalizable over $\mathbb{R}$. *Find* $\mathbf{P}$ *such that* $\mathbf{P}^{-1}AP$ *is diagonal and hence find* $\mathbf{A}^{25}$.

Solution. Characteristic equation of $\mathbf{A}$ is

$$\begin{aligned} \begin{vmatrix} 2-x & -1 & 0 \\ -1 & 2-x & 0 \\ 2 & 2 & 3-x \end{vmatrix} &= 0 \\ \Rightarrow \quad (2-x)(2-x)(3-x) + 1(-(3-x)) &= 0 \\ (3-x)(4-4x+x^2-1) &= 0 \end{aligned}$$

Thus the eigenvalues are 3, 3, 1.

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 1$.

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $x_1 - x_2 = 0, -x_1 + x_2 = 0, 2x_1 + 2x_2 + 2x_3 = 0$. Take $x_1 = 1$, then $x_2 = 1, x_3 = -2$, so $(1, 1, -2)$ is an eigenvector with eigenvalue 1.

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 3$.

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $x_1 + x_2 = 0$. Take $x_1 = 1, x_3 = 0$, then $x_2 = -1$, so $(1, -1, 0)$ is an eigenvector with eigenvalue 3. Take $x_1 = 0, x_3 = 1$, then $x_2 = 0$, so $(0, 0, 1)$ is also an eigenvector for eigenvalue 3.

Let $\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ then $\mathbf{AP} = \mathbf{P} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ or $\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Now $\mathbf{P}^{-1}\mathbf{A}^{25}\mathbf{P} = (\mathbf{P}^{-1}\mathbf{AP})^{25} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^{25} & 0 \\ 0 & 0 & 3^{25} \end{pmatrix}$. Thus $\mathbf{A}^{25} = \mathbf{P} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^{25} & 0 \\ 0 & 0 & 3^{25} \end{pmatrix} \mathbf{P}^{-1}$

$|\mathbf{P}| = -2$, so $\mathbf{P}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

$$
\begin{aligned}
\mathbf{A}^{25} &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^{25} & 0 \\ 0 & 0 & 3^{25} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 3^{25} & 0 \\ 1 & -3^{25} & 0 \\ -2 & 0 & 3^{25} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1+3^{25}}{2} & \frac{1-3^{25}}{2} & 0 \\ \frac{1-3^{25}}{2} & \frac{1+3^{25}}{2} & 0 \\ -1+3^{25} & -1+3^{25} & 3^{25} \end{pmatrix}
\end{aligned}
$$

∎

Question 2(c) *Let $\mathbf{A} = [a_{ij}]$ be a square matrix of order n such that $|a_{ij}| \leq M$. Let λ be an eigenvalue of $\mathbf{A}$, show that $|\lambda| \leq nM$.*

Solution. We first prove the following:

Lemma: If $\mathbf{A} = [a_{ij}]$ and $\sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \leq a_{ii}$ then $|\mathbf{A}| \neq 0$.

If $|\mathbf{A}| = 0$ then there exist $x_1, \ldots, x_n \in \mathbb{C}$ not all zero such that

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\
&\ldots \\
a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n &= 0 \\
&\ldots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= 0
\end{aligned}
$$

Let $|x_i| = \max(|x_1|, |x_2|, \ldots, |x_n|)$, so $|\frac{x_j}{x_i}| \leq 1$ for all j.

$$
\begin{aligned}
0 &= \left| a_{ii} - (-a_{i1}\frac{x_1}{x_i} - a_{i2}\frac{x_2}{x_i} - \ldots - a_{in}\frac{x_n}{x_i}) \right| \\
&\geq |a_{ii}| - \left| a_{i1}\frac{x_1}{x_i} + a_{i2}\frac{x_2}{x_i} + \ldots + a_{in}\frac{x_n}{x_i} \right| \\
&\geq |a_{ii}| - |a_{i1}| - |a_{i2}| - \ldots - |a_{in}|
\end{aligned}
$$

which contradicts the premise $\sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \leq a_{ii}$ Thus $|\mathbf{A}| \neq 0$.

Now the lemma tells us that if $|\lambda - a_{ii}| > \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}|$ then $|\lambda \mathbf{I} - \mathbf{A}| \neq 0$, so λ is not an eigenvalue of $\mathbf{A}$. Thus $|\lambda| \leq |\lambda - a_{ii}| + |a_{ii}| \leq \sum_{j=1}^{n} |a_{ij}| \leq nM$ as desired.

∎

Question 3(a) *Define a positive definite matrix and show that a positive definite matrix is always non-singular. Show that the converse is not always true.*

Solution. Let $\mathbf{A}$ be an $n \times n$ real symmetric matrix. $\mathbf{A}$ is said to be positive definite if the associated quadtratic form

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} > 0$$

for all $(x_1, x_2, \dots, x_n) \neq (0, 0, \dots, 0)$ in $\mathbb{R}^n$.

If $|\mathbf{A}| = 0$ then rank $\mathbf{A} < n$, which means that columns of $\mathbf{A}$ i.e. $\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}$ are linearly dependent i.e. there exist real numbers $x_1, x_2, \dots, x_n$ not all zero such that

$$x_1\mathbf{c_1} + x_2\mathbf{c_2} + \dots + x_n\mathbf{c_n} = \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \mathbf{0} \Longrightarrow \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = 0$$

where $(x_1, x_2, \dots, x_n) \neq (0, 0, \dots, 0)$, which means that $\mathbf{A}$ is not positive definite. Thus $\mathbf{A}$ is positive definite $\Longrightarrow |\mathbf{A}| \neq 0$.

The converse is not true. Take

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

then $|\mathbf{A}| = -1$, but

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -1$$

so $\mathbf{A}$ is not positive definite.

∎

Question 3(b) *Find the eigenvalues and their corresponding eigenvectors for the matrix*

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

Solution. The characteristic equation for $\mathbf{A}$ is

$$\begin{aligned}
0 &= |\mathbf{A} - x\mathbf{I}| \\
&= \begin{pmatrix} 6-x & -2 & 2 \\ -2 & 3-x & -1 \\ 2 & -1 & 3-x \end{pmatrix} \\
&= (6-x)((3-x)^2 - 1) + 2(-6+2x+2) + 2(2-6+2x) \\
&= (6-x)(9-6x+x^2) - 6 + x - 8 + 4x - 8 + 4x \\
0 &= x^3 - 12x^2 + 36x - 32 \\
&= (x-2)(x^2 - 10x + 16)
\end{aligned}$$

Thus the eigenvalues are 2, 2, 8.

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 2$.

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $4x_1 - 2x_2 + 2x_3 = 0, -2x_1 + x_2 - x_3 = 0, 2x_1 - x_2 + x_3 = 0$. Take $x_1 = 1, x_2 = 0$, then $x_3 = -2$, so $(1, 0, -2)$ is an eigenvector with eigenvalue 2. Take $x_1 = 0, x_2 = 1$, then $x_3 = 1$, so $(0, 1, 1)$ is an eigenvector with eigenvalue 2.

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 8$.

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $-2x_1 - 2x_2 + 2x_3 = 0, -2x_1 - 5x_2 - x_3 = 0, 2x_1 - x_2 - 5x_3 = 0$. From the last two, we get $x_2 + x_3 = 0$, and from the first we get $x_1 = 2x_3$. Take $x_3 = 1$, then $x_2 = -1, x_1 = 2$, so $(2, -1, 1)$ is an eigenvector with eigenvalue 8. ∎

Question 3(c) *Find* $\mathbf{P}$ *invertible such that* $\mathbf{P}$ *reduces* $Q(x, y, z) = 2xy + 2yz + 2zx$ *to its canonical form.*

Solution. The matrix of $Q(x, y, z)$ is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

which has all diagonal entries 0, so we cannot complete squares right away.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Add the second row to the first and the second column to the first.

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Subtract $\frac{1}{2}R_1$ from R_2 and $\frac{1}{2}C_1$ from C_2.

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & -\frac{1}{2} & 0 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Subtract R_1 from R_3 and C_1 from C_3.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -1 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 1 & \frac{1}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus $\mathbf{P} = \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 1 & \frac{1}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{P}'\mathbf{A}\mathbf{P} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -2 \end{pmatrix}$

So $Q(x,y,z) \longrightarrow 2X^2 - \frac{1}{2}Y^2 - 2Z^2$.

Alternative Solution. Let $x = X, y = X + Y, z = Z$

$$\begin{aligned} Q(x,y,z) &= 2X^2 + 2XY + 2ZX + 2ZY + 2ZX \\ &= 2[X^2 + XY + 2ZX + ZY] \\ &= 2[(X + \frac{Y}{2} + Z)^2 - \frac{Y^2}{4} - Z^2] \end{aligned}$$

Put $\xi = X + Y/2 + Z, \eta = Y, \zeta = Z$, so $X = \xi - \eta/2 - \zeta, Y = \eta, Z = \zeta$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 1 & \frac{1}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

Thus $Q(x,y,z) \longrightarrow 2\xi^2 - \eta^2/2 - 2\zeta^2$, and $\mathbf{P} = \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 1 & \frac{1}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix}$ as before. Note that we put $x = X, y = X + Y, z = Z$ to create one square term to complete the squares. ∎