UPSC Civil Services Main 1999 - Mathematics Linear Algebra

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June 14, 2007

Question 1(a) *Let $\mathcal{V}$ be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $f, g, h \in \mathcal{V}$ are linearly independent where $f(t) = e^{2t}$, $g(t) = t^2$ and $h(t) = t$.*

Solution. Let $a, b, c \in \mathbb{R}$ and let $ae^{2t} + bt^2 + ct = 0$ for all t. Setting $t = 0$ shows that $a = 0$. From $t = 1$ we get $b + c = 0$, and $t = -1$ gives $b - c = 0$, hence $b = c = 0$. Thus f, g, h are linearly independent. ∎

Question 1(b) *If the matrix of the linear transformation $\mathbf{T}$ on $\mathcal{V}_2(\mathbb{R})$ with respect to the basis $B = \{(1,0),(0,1)\}$ is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then what is the matrix of $\mathbf{T}$ with respect to the ordered basis $B_1 = \{(1,1),(1,-1)\}$.*

Solution. $\mathbf{T}(\mathbf{e_1}) = \mathbf{e_1} + \mathbf{e_2}, \mathbf{T}(\mathbf{e_2}) = \mathbf{e_1} + \mathbf{e_2}$. Let $\mathbf{v_1} = (1,1) = \mathbf{e_1} + \mathbf{e_2}, \mathbf{v_2} = (1,-1) = \mathbf{e_1} - \mathbf{e_2}$. Then $\mathbf{T}(\mathbf{v_1}) = \mathbf{T}((1,1)) = (2,2) = 2\mathbf{e_1} + 2\mathbf{e_2} = 2\mathbf{v_1}$. $\mathbf{T}(\mathbf{v_2}) = \mathbf{T}((1,-1)) = (0,0) = \mathbf{0}$. Thus the matrix of $\mathbf{T}$ with respect to the ordered basis $B_1 = \{(1,1),(1,-1)\}$ is $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. ∎

Question 1(c) *Diagonalize the matrix* $\mathbf{A} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$.

Solution. The characteristic equation is

$$\begin{aligned}
0 &= \begin{vmatrix} 4-x & 2 & 2 \\ 2 & 4-x & 2 \\ 2 & 2 & 4-x \end{vmatrix} \\
&= (4-x)((4-x)^2-4) + 2(4-8+2x) - 2(8-2x-4) \\
&= (4-x)(12-8x+x^2) - 8 + 4x - 8 + 4x \\
&= 48 - 32x + 4x^2 - 12x + 8x^2 - x^3 - 16 + 8x \\
&= -(x^3 - 12x^2 + 36x - 32) \\
&= -(x-2)(x^2-10x+16) \quad \because\ 2 \text{ is a root} \\
&= -(x-2)(x-2)(x-8)
\end{aligned}$$

The characteristic roots are 2, 2, 8.

If (x_1, x_2, x_3) is an eigenvector for $\lambda = 8$, then

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus $x_1 = x_2 = x_3$, so $(1, 1, 1)$ is an eigenvector for $\lambda = 8$.

Similarly for $\lambda = 2$,

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus $x_1 + x_2 + x_3 = 0$. Take $x_1 = 1, x_2 = 0$, so $(1, 0, -1)$ is an eigenvector. Take $x_1 = 0, x_2 = 1$, so $(0, 1, -1)$ is an eigenvector for $\lambda = 2$. These eigenvectors are linearly independent. Thus if $\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}$, then $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

To check, verify that $\mathbf{AP} = \mathbf{P} \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ ∎

Question 2(a) *Test for congruency the matrices* $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* $\mathbf{B} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. *Prove that* $\mathbf{A}^{2n} = \mathbf{B}^{2m} = \mathbf{I}$ *where* m, n *are positive integers.*

Solution. $\mathbf{A}$ and $\mathbf{B}$ are not congruent, because $\mathbf{A}$ is symmetric and $\mathbf{B}$ is not. If $\mathbf{A} \equiv \mathbf{B}$ then $\exists \mathbf{P}$ non-singular such that $\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{B}$ which implies that $\mathbf{B}$ should be symmetric.

$$\begin{aligned}
\mathbf{A}^2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\mathbf{B}^2 &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Hence $\mathbf{A}^{2n} = (\mathbf{A}^2)^n = \mathbf{I}$, and $\mathbf{B}^{2m} = (\mathbf{B}^2)^m = \mathbf{I}$. ∎

Question 2(b) *If* $\mathbf{A}$ *is a skew symmetric matrix of order* n *then prove that* $(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1}$ *is orthogonal.*

Solution.

$$\begin{aligned}
\mathbf{O} &= (\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1} \\
\mathbf{O}\mathbf{O}' &= (\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1}((\mathbf{I}+\mathbf{A})^{-1})'(\mathbf{I}-\mathbf{A})' \\
&= (\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1}(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{I}+\mathbf{A}) \quad \textit{as } \mathbf{A}' = -\mathbf{A} \\
&= (\mathbf{I}-\mathbf{A})[(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})]^{-1}(\mathbf{I}+\mathbf{A}) \\
&= (\mathbf{I}-\mathbf{A})[\mathbf{I}-\mathbf{A}^2)]^{-1}(\mathbf{I}+\mathbf{A}) \\
&= (\mathbf{I}-\mathbf{A})[(\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})]^{-1}(\mathbf{I}+\mathbf{A}) \\
&= (\mathbf{I}-\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{I}+\mathbf{A})^{-1}(\mathbf{I}+\mathbf{A}) \\
&= \mathbf{I}
\end{aligned}$$

Similarly it can be shown that $\mathbf{O}'\mathbf{O} = \mathbf{I}$. Hence $\mathbf{O}$ is orthogonal. ∎

Question 2(c) *Test for positive definiteness the quadratic form* $2x^2 + y^2 + 2z^2 + 2xy - 2zx$.

Solution. The given form is
$\frac{1}{2}(4x^2 + 3y^2 + 4z^2 + 4xy - 4zx)$
$= \frac{1}{2}((2x + y - z)^2 + y^2 + 3z^2 + 2yz)$
$= \frac{1}{2}((2x + y - z)^2 + (y + z)^2 + 2z^2)$
Now $2x + y - z = 0, y + z = 0, z = 0$ implies $x = y = z = 0$. Hence the form is positive definite. ∎

Question 2(d) *Reduce the equation*

$$x^2 + y^2 + z^2 - 2xy - 2yz + 2zx + x - y - 2z + 6 = 0$$

into canonical form and determine the nature of the quadric.

Solution. Consider $x^2 + y^2 + z^2 - 2xy - 2yz + 2zx$. Its matrix is

$$\mathbf{S} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

Its characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

or $\lambda^3 - 3\lambda^2 = 0$. Thus $\lambda = 0, 0, 3$.

We next determine the characteristic vectors. For $\lambda = 0$, we get

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus $x_1 - x_2 + x_3 = 0$. Take $(1, 1, 0)$ and $(-1, 1, 2)$ as orthogonal characteristic vectors corresponding to $\lambda = 0$.

For $\lambda = 3$, we get

$$\begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This yields $x_1 = -x_2 = x_3$. Take $(1, -1, 1)$ as the characteristic vector for $\lambda = 3$.

Thus if

$$\mathbf{O} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

Then $\mathbf{O'SO} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Let

$$\mathbf{O} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

or

$$\begin{aligned} x &= \frac{X}{\sqrt{3}} + \frac{Y}{\sqrt{2}} - \frac{Z}{\sqrt{6}} \\ y &= -\frac{X}{\sqrt{3}} + \frac{Y}{\sqrt{2}} + \frac{Z}{\sqrt{6}} \\ z &= \frac{X}{\sqrt{3}} + \frac{2Z}{\sqrt{6}} \end{aligned}$$

Thus the given equation can be transformed to

$$3X^2 + \tfrac{X}{\sqrt{3}} + \tfrac{Y}{\sqrt{2}} - \tfrac{Z}{\sqrt{6}} + \tfrac{X}{\sqrt{3}} - \tfrac{Y}{\sqrt{2}} - \tfrac{Z}{\sqrt{6}} - \tfrac{2X}{\sqrt{3}} - \tfrac{4Z}{\sqrt{6}} + 6 = 0$$
$$\Rightarrow 3X^2 - Z\sqrt{6} + 6 = 0$$
$$\Rightarrow \sqrt{\tfrac{3}{2}}X^2 = Z - \sqrt{6}$$

Shifting the origin to $(0, 0, \sqrt{6})$, we get $X^2 = \sqrt{\frac{2}{3}}Z$, showing that the equation is a parabolic cylinder. ∎

1 Reduction of Quadrics

For the sake of completeness, we give the complete theoretical discussion for the above question.

Let

$$F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

It can be expressed in matrix form as

$$\begin{pmatrix} x & y & z & 1 \end{pmatrix} \begin{pmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0$$

Let the 4×4 matrix be $\mathbf{Q}$.

Step I. Consider

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix} \mathbf{S} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

1. Find the characteristic roots of $\mathbf{S}$ — $\lambda_1, \lambda_2, \lambda_3$.

2. Find characteristic vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_2}$ corresponding to $\lambda_1, \lambda_2, \lambda_3$ which are orthogonal. These on normalization give us

$$\mathbf{O} = \begin{pmatrix} \frac{\mathbf{v_1}}{||\mathbf{v_1}||} & \frac{\mathbf{v_2}}{||\mathbf{v_2}||} & \frac{\mathbf{v_3}}{||\mathbf{v_3}||} \end{pmatrix}$$

Thus we get $\mathbf{O'SO}$ =diagonal$(\lambda_1, \lambda_2, \lambda_3)$. Let

$$\mathbf{O} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

This gives us three equations expressing x, y, z in term of X, Y, Z. Substituting in F, we get

$$F(X, Y, Z) = \lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 + 2UX + 2VY + 2WZ + d = 0$$

(Note that d is unaffected.)

Note 1 *Since* $\mathbf{O}$ *is orthogonal, the transformation* $\mathbf{O} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ *is just a rotation of the axes, and therefore the nature of the quadric is unaffected.*

Step II. We now consider 3 possibilities (ρ is rank of the matrix):

1. $\rho(\mathbf{S}) = 3 \Rightarrow \lambda_1\lambda_2\lambda_3 \neq 0$. Shift the origin to $(-\frac{U}{\lambda_1}, -\frac{V}{\lambda_2}, -\frac{W}{\lambda_3})$, i.e. $x = X + \frac{U}{\lambda_1}, y = Y + \frac{V}{\lambda_2}, z = Z + \frac{W}{\lambda_3}$. (Actually we are just completing the squares.) F gets transformed to
$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d_2 = 0$$
2. $\rho(\mathbf{S}) = 2$. One characteristic root, say $\lambda_3 = 0$. Shift the origin to $(-\frac{U}{\lambda_1}, -\frac{V}{\lambda_2}, 0)$, and F gets transformed to
$$\lambda_1 x^2 + \lambda_2 y^2 + 2w_2 z + d_2 = 0$$
3. $\rho(\mathbf{S}) = 1$. Two characteristic roots, say $\lambda_2 = \lambda_3 = 0$. Shift the origin to $(-\frac{U}{\lambda_1}, 0, 0)$, and F gets transformed to
$$\lambda_1 x^2 + 2v_2 y + 2w_2 z + d_2 = 0$$

Note that $\rho(\mathbf{S}) = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$. Then $F(x, y, z)$ is no longer a quadric, it is a plane.

Step III. Observe that $\rho(\mathbf{S}) \leq \rho(\mathbf{Q}) \leq 4, \rho(\mathbf{S}) \leq 3$.

1. Let $\rho(\mathbf{Q}) = 4, \rho(\mathbf{S}) = 3$. As shown above, $F(x, y, z) = 0$ is transformed to
$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d_2 = 0$$
$|\mathbf{Q}| = \lambda_1\lambda_2\lambda_3 d_2 \Rightarrow d_2 = \frac{|\mathbf{Q}|}{|\mathbf{S}|}$. Thus the quadric is $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = -\frac{|\mathbf{Q}|}{|\mathbf{S}|}$, which is a central quadric i.e. a quadric surface with a center, e.g., a sphere, ellipsoid, or hyperboloid, depending upon the signs and magnitudes of the eigenvalues. If the right hand side has positive sign (maybe by multiplying the equation with -1), then look at the signs of the coefficients of the l.h.s. If all are positive, it is an ellipsoid, further if all are equal, it is a sphere. If 1 or 2 are negative, it is a hyperboloid. If all 3 are negative, the surface is the empty set.
2. $\rho(\mathbf{Q}) = 3, \rho(\mathbf{S}) = 3$. $|\mathbf{Q}| = \lambda_1\lambda_2\lambda_3 d_2 = 0 \Rightarrow d_2 = 0$, because $\lambda_1\lambda_2\lambda_3 \neq 0$. Thus the quadric becomes $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0$, which is a cone.
3. $\rho(\mathbf{Q}) = 4, \rho(\mathbf{S}) = 2$
$$F(x, y, z) = \lambda_1 x^2 + \lambda_2 y^2 + 2w_2 z + d_2 = 0$$
$$\mathbf{Q} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & w_2 \\ 0 & 0 & w_2 & d_2 \end{pmatrix}$$
$|\mathbf{Q}| = -\lambda_1\lambda_2 w_2^2$. Since $\rho(\mathbf{Q}) = 4$, $w_2 \neq 0$. Shifting the origin to $(0, 0, -\frac{d_2}{2w_2})$ we get
$$F(x, y, z) = \lambda_1 x^2 + \lambda_2 y^2 + 2w_2 z = 0$$
where $w_2^2 = -|\mathbf{Q}|/\lambda_1\lambda_2$. The surface is a paraboloid.

4. $\rho(\mathbf{Q}) = 3, \rho(\mathbf{S}) = 2$

$$F(x, y, z) = \lambda_1 x^2 + \lambda_2 y^2 + 2w_2 z + d_2 = 0$$

$\rho(\mathbf{Q}) = 3 \Rightarrow |\mathbf{Q}| = -\lambda_1 \lambda_2 w_2^2 \Rightarrow w_2 = 0$. Since

$$\mathbf{Q} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2 \end{pmatrix}$$

and $\rho(\mathbf{Q}) = 3$, $d_2 \neq 0$. Thus

$$F(x, y, z) = \lambda_1 x^2 + \lambda_2 y^2 + d_2 = 0$$

The quadric is a hyperbolic or elliptic cylinder.

5. $\rho(\mathbf{Q}) = 2, \rho(\mathbf{S}) = 2$

$$\mathbf{Q} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2 \end{pmatrix}$$

$\rho(\mathbf{Q}) = 2 \Rightarrow d_2 = 0$, and $F(x, y, z) = \lambda_1 x^2 + \lambda_2 y^2 = 0$. The quadric is a pair of distinct planes or a point, if $\lambda_1 = \lambda_2 \neq 0$.

6. $\rho(\mathbf{Q}) = 4, \rho(\mathbf{S}) = 1$

$$F(x, y, z) = \lambda_1 x^2 + 2v_2 y + 2w_2 z + d_2 = 0$$

$$\mathbf{Q} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_2 \\ 0 & 0 & 0 & w_2 \\ 0 & v_2 & w_2 & d_2 \end{pmatrix}$$

which shows that $\rho(\mathbf{Q}) = 4$ is not possible.

7. $\rho(\mathbf{Q}) = 3, \rho(\mathbf{S}) = 1$

$$F(x, y, z) = \lambda_1 x^2 + 2v_2 y + 2w_2 z + d_2 = 0$$

$\rho(\mathbf{Q}) = 3 \Rightarrow$ both v_2 and w_2 cannot be 0. Suppose $v_2 \neq 0$. Shift the origin to $(0, -\frac{d_2}{2v_2}, 0)$.

$$F(x, y, z) = \lambda_1 x^2 + 2v_2 y + 2w_2 z = 0$$

Rotate the axes by

$$\begin{aligned} x &= X \\ y &= \frac{v_2}{\sqrt{v_2^2+w_2^2}}Y - \frac{w_2}{\sqrt{v_2^2+w_2^2}}Z \\ z &= \frac{w_2}{\sqrt{v_2^2+w_2^2}}Y + \frac{v_2}{\sqrt{v_2^2+w_2^2}}Z \end{aligned}$$

$$F(x,y,z) = \lambda_1 X^2 + 2v_3 Y = 0 \qquad v_3 = \sqrt{v_2^2+w_2^2}$$

Thus the quadric is a parabolic cylinder.

8. $\rho(\mathbf{Q}) = 2, \rho(\mathbf{S}) = 1$ $\rho(\mathbf{Q}) = 2 \Rightarrow v_2 = w_2 = 0, d_2 \neq 0.$

$$F(x,y,z) = \lambda_1 X^2 + d_2 = 0$$

The quadric is two parallel planes.

9. $\rho(\mathbf{Q}) = 1, \rho(\mathbf{S}) = 1$ The quadric immediately reduces to $F(x,y,z) = \lambda_1 X^2 = 0$, so it represents two coincident planes $x = 0$.

$\rho(\mathbf{Q})$	$\rho(\mathbf{S})$	Surface	Canonical form
4	3	central quadric	$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = -\frac{\lvert\mathbf{Q}\rvert}{\lvert\mathbf{S}\rvert}$
3	3	cone	$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0$
4	2	paraboloid	$\lambda_1 x^2 + \lambda_2 y^2 + 2w_2 z = 0, w_2^2 = -\frac{\lvert\mathbf{Q}\rvert}{\lambda_1\lambda_2}$
3	2	elliptic or hyperbolic cylinder	$\lambda_1 x^2 + \lambda_2 y^2 + d_2 = 0$
2	2	pair of distinct planes if $\lambda_1\lambda_2 < 0$ point if $\lambda_1\lambda_2 > 0$	$\lambda_1 x^2 + \lambda_2 y^2 = 0$
4	1	Not possible	
3	1	parabolic cylinder	$\lambda_1 X^2 + 2v_3 Y = 0, v_3 = \sqrt{v_2^2 + w_2^2}$
2	1	pair of parallel planes	$\lambda_1 X^2 + d_2 = 0$
1	1	Two coincident planes	$\lambda_1 X^2 = 0$