UPSC Civil Services Main 2000 - Mathematics Linear Algebra

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Question 1(a) *Let $\mathcal{V}$ be a vector space over $\mathbb{R}$ and let*

$$\mathcal{T} = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in \mathcal{V}\}$$

Define $(\mathbf{x}, \mathbf{y}) + (\mathbf{x_1}, \mathbf{y_1}) = (\mathbf{x} + \mathbf{x_1}, \mathbf{y} + \mathbf{y_1})$ in $\mathcal{T}$ and $(\alpha + i\beta)(\mathbf{x}, \mathbf{y}) = (\alpha\mathbf{x} - \beta\mathbf{y}, \beta\mathbf{x} + \alpha\mathbf{y})$ for every $\alpha, \beta \in \mathbb{R}$. Show that $\mathcal{T}$ is a vector space over $\mathbb{C}$.

Solution.

1. $\mathbf{v_1}, \mathbf{v_2} \in \mathcal{T} \Rightarrow \mathbf{v_1} + \mathbf{v_2} \in \mathcal{T}$
2. $(\mathbf{0}, \mathbf{0})$ is the additive identity where $\mathbf{0}$ is the zero vector in $\mathcal{V}$.
3. If $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}$, then $(-\mathbf{x}, -\mathbf{y}) \in \mathcal{T}$, and $(\mathbf{x}, \mathbf{y}) + (-\mathbf{x}, -\mathbf{y}) = (\mathbf{0}, \mathbf{0})$
4. Clearly $\mathbf{v_1} + \mathbf{v_2} = \mathbf{v_2} + \mathbf{v_1}$ and $(\mathbf{v_1} + \mathbf{v_2}) + \mathbf{v_3} = \mathbf{v_1} + (\mathbf{v_2} + \mathbf{v_3})$ as addition is commutative and associative in $\mathcal{V}$.
5. $z \in \mathbb{C}, \mathbf{v} \in \mathcal{T} \Rightarrow z\mathbf{v} \in \mathcal{T}$
6. $1\mathbf{v} = (1 + i0)(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{y})$
7.

$$\begin{aligned} & (\alpha + i\beta)((\mathbf{x_1}, \mathbf{y_1}) + (\mathbf{x_2}, \mathbf{y_2})) \\ = \; & (\alpha(\mathbf{x_1} + \mathbf{x_2}) - \beta(\mathbf{y_1} + \mathbf{y_2}), \beta(\mathbf{x_1} + \mathbf{x_2}) + \alpha(\mathbf{y_1} + \mathbf{y_2})) \\ = \; & (\alpha\mathbf{x_1} - \beta\mathbf{y_1}, \beta\mathbf{x_1} + \alpha\mathbf{y_1}) + (\alpha\mathbf{x_2} - \beta\mathbf{y_2}, \beta\mathbf{x_2} + \alpha\mathbf{y_2}) \\ = \; & (\alpha + i\beta)(\mathbf{x_1}, \mathbf{y_1}) + (\alpha + i\beta)(\mathbf{x_2}, \mathbf{y_2}) \end{aligned}$$

8.

$$\begin{aligned}
& ((\alpha + i\beta)(\gamma + i\delta))(\mathbf{x}, \mathbf{y}) \\
= \;& (\alpha\gamma - \beta\delta + i(\alpha\delta + \beta\gamma))(\mathbf{x}, \mathbf{y}) \\
= \;& ((\alpha\gamma - \beta\delta)\mathbf{x} - (\alpha\delta + \beta\gamma)\mathbf{y}, (\alpha\gamma - \beta\delta)\mathbf{y} + (\alpha\delta + \beta\gamma)\mathbf{x}) \\
= \;& (\alpha(\gamma\mathbf{x} - \delta\mathbf{y}) - \beta(\delta\mathbf{x} + \gamma\mathbf{y}), \beta(\gamma\mathbf{x} - \delta\mathbf{y}) + \alpha(\delta\mathbf{x} + \gamma\mathbf{y})) \\
= \;& (\alpha + i\beta)((\gamma\mathbf{x} - \delta\mathbf{y}), (\delta\mathbf{x} + \gamma\mathbf{y})) \\
= \;& (\alpha + i\beta)((\gamma + i\delta)(\mathbf{x}, \mathbf{y}))
\end{aligned}$$

Thus $\mathcal{T}$ is a vector space over $\mathbb{C}$. ∎

Question 1(b) *Show that if λ is a characteristic root of a non-singular matrix $\mathbf{A}$, then λ^{-1} is a characteristic root of $\mathbf{A}^{-1}$.*

Solution.

$$\begin{aligned}
\mathbf{A}\mathbf{v} &= \lambda\mathbf{v} \qquad \mathbf{v} \neq \mathbf{0} \\
\Rightarrow \quad \mathbf{A}^{-1}\mathbf{A}\mathbf{v} &= \lambda\mathbf{A}^{-1}\mathbf{v} \\
\Rightarrow \quad \mathbf{A}^{-1}\mathbf{v} &= \lambda^{-1}\mathbf{v}
\end{aligned}$$

Thus λ^{-1} is a characteristic root of $\mathbf{A}^{-1}$. ∎

Question 2(a) *Prove that a real symmetric matrix $\mathbf{A}$ is positive definite if and only if $\mathbf{A} = \mathbf{B}\mathbf{B}'$ for some non-singular $\mathbf{B}$. Show also that $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 11 \end{pmatrix}$ is positive definite and find $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}\mathbf{B}'$. (Here $\mathbf{B}'$ is the transpose of $\mathbf{B}$.)*

Solution. If $\mathbf{A} = \mathbf{B}\mathbf{B}'$ for some non-singular $\mathbf{B}$, then $\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{B}\mathbf{B}'\mathbf{x}$ where $\mathbf{x} \neq \mathbf{0}$ is a column vector. Since $|\mathbf{B}| \neq 0$, $\mathbf{B}'\mathbf{x} \neq \mathbf{0} \Longrightarrow \mathbf{x}'\mathbf{B}.(\mathbf{B}'\mathbf{x})$ is the sum on n squares, at least one of which is non-zero. Thus $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$ whenever $\mathbf{x} \neq \mathbf{0}$, showing that $\mathbf{A}$ is positive definite.

Conversely, if $\mathbf{A}$ is positive definite, then $\exists\mathbf{P}$ non-singular such that $\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{I_n}$. Thus $\mathbf{A} = \mathbf{P}'^{-1}\mathbf{P}^{-1}$. Letting $\mathbf{B} = \mathbf{P}'^{-1}$ we get $\mathbf{A} = \mathbf{B}\mathbf{B}'$ as required.

The existence of $\mathbf{P}$ can be found by induction on n. Let

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \dots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Define

$$\mathbf{Q} = \begin{pmatrix} 1 & -\frac{a_{12}}{a_{11}} & \dots & -\frac{a_{1n}}{a_{11}} \\ 0 & 1 & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Then $\mathbf{Q}$ is non-singular, and $\mathbf{Q'AQ} = \begin{pmatrix} a_{11} & 0 \\ 0 & \mathbf{S} \end{pmatrix}$, where $\mathbf{S}$ is $(n-1) \times (n-1)$ positive definite. Let $\mathbf{Q}^*$ be a $(n-1) \times (n-1)$ non-singular matrix such that $\mathbf{Q}^{*\prime}\mathbf{S}\mathbf{Q}^*$ is diagonal, by induction. Then let $\mathbf{Q_1} = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{Q}^* \end{pmatrix}$, and let $\mathbf{P} = \mathbf{Q_1 Q}$. Then $\mathbf{P'AP}$ is diagonal $(b_{11}, b_{22}, \ldots, b_{nn})$. Let $\mathbf{B} =$ diagonal $(\frac{1}{\sqrt{b_{11}}}, \ldots, \frac{1}{\sqrt{b_{nn}}})$. Then $\mathbf{B'P'APB} = \mathbf{I_n}$.

The quadratic form $Q(x, y, z)$ associated with the given matrix $\mathbf{A}$ is given by

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + 5y^2 + 11z^2 + 4xy + 6xz + 14yz$$

Completing the squares we get $Q(x, y, z) = (x + 2y + 3z)^2 + (y + z)^2 + z^2$, so $\mathbf{A}$ is positive definite, as $z = 0, y + z = 0, x + 2y + 3z = 0 \Longrightarrow x = y = z = 0$.

If $\mathbf{B}$ is a 3×3 matrix such that

$$\mathbf{B}' \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ y + z \\ z \end{pmatrix}$$

then $\mathbf{x'BB'x} = Q = \mathbf{x'Ax}$, so $\mathbf{A} = \mathbf{BB'}$ as $\mathbf{A}$ and $\mathbf{BB'}$ are both symmetric. Clearly

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}$$

and it can easily be verified that $\mathbf{A} = \mathbf{BB'}$. ∎

Question 2(b) *Prove that a system* $\mathbf{Ax} = \mathbf{B}$ *of non-homogeneous equations in* n *unknowns has a unique solution provided the coefficient matrix is non-singular.*

Solution. If $\mathbf{A}$ is non-singular, then the system is consistent because the rank of the coefficient matrix $\mathbf{A} = n =$ rank of the $n \times n+1$ augmented matrix $(\mathbf{A}, \mathbf{B})$. If $\mathbf{x_1}, \mathbf{x_2}$ are two solutions, then

$$\begin{aligned} & \mathbf{Ax_1} = \mathbf{B} = \mathbf{Ax_2} \\ \Longrightarrow\ & \mathbf{A}(\mathbf{x_1} - \mathbf{x_2}) = \mathbf{0} \\ \Longrightarrow\ & \mathbf{A}^{-1}\mathbf{A}(\mathbf{x_1} - \mathbf{x_2}) = \mathbf{0} \\ \Longrightarrow\ & \mathbf{x_1} = \mathbf{x_2} \end{aligned}$$

Thus the unique solution is given by the column vector $\mathbf{x} = \mathbf{A}^{-1}\mathbf{B}$.

∎

Question 2(c) *Prove that two similar matrices have the same characteristic roots. Is the converse true? Justify your claim.*

Solution. Let $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$ then characteristic polynomial of $\mathbf{B}$ is $|\lambda\mathbf{I} - \mathbf{B}| = |\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}| = |\mathbf{P}^{-1}\lambda\mathbf{IP} - \mathbf{P}^{-1}\mathbf{AP}| = |\mathbf{P}^{-1}||\lambda\mathbf{I} - \mathbf{A}||\mathbf{P}| = |\lambda\mathbf{I} - \mathbf{A}|$. (Note that $|\mathbf{X}||\mathbf{Y}| = |\mathbf{XY}|$.) Thus the characteristic polynomial of $\mathbf{B}$ is the same as that of $\mathbf{A}$, so both $\mathbf{A}$ and $\mathbf{B}$ have the same characteristic roots.

The converse is not true. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then $\mathbf{A}$ and $\mathbf{B}$ have the same characteristic polynomial $(\lambda - 1)^2$ and thus the same characteristic roots. But $\mathbf{B}$ can never be similar to $\mathbf{A}$ because $\mathbf{P}^{-1}\mathbf{BP} = \mathbf{B}$ whatever $\mathbf{P}$ may be. ∎