UPSC Civil Services Main 2002 - Mathematics Linear Algebra

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Question 1(a) *Show that the mapping* $\mathbf{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *where* $\mathbf{T}(a, b, c) = (a - b, b - c, a + c)$ *is linear and non-singular.*

Solution.

$$\begin{aligned} \mathbf{T}(\lambda a, \lambda b, \lambda c) &= (\lambda(a-b), \lambda(b-c), \lambda(a+c)) \\ &= \lambda(a-b, b-c, a+c) \\ &= \lambda \mathbf{T}(a, b, c) \end{aligned}$$

$$\begin{aligned} \mathbf{T}(a, b, c) + \mathbf{T}(x, y, z) &= (a-b, b-c, a+c) + (x-y, y-z, x+z) \\ &= (a-b+x-y, b-c+y-z, a+c+x+z) \\ &= \mathbf{T}(a+x, b+y, c+z) \end{aligned}$$

Thus $\mathbf{T}$ is linear.

Now we show that

$$\begin{aligned} \mathbf{T}(1,0,0) &= (1,0,1) \\ \mathbf{T}(0,1,0) &= (-1,1,0) \\ \mathbf{T}(0,0,1) &= (0,-1,1) \end{aligned}$$

are linearly independent.

$$\begin{aligned} & a_1(1,0,1) + a_2(-1,1,0) + a_3(0,-1,1) = 0 \\ \Rightarrow\ & a_1 - a_2 = 0, a_2 - a_3 = 0, a_1 + a_3 = 0 \\ \Rightarrow\ & a_1 = a_2 = a_3 = 0 \end{aligned}$$

Thus $(1,0,1),(-1,1,0),(0,-1,1)$ are linearly independent.

Since $(1,0,0),(0,1,0),(0,0,1)$ generate $\mathbb{R}^3$, $(1,0,1),(-1,1,0),(0,-1,1)$ generate $\mathbf{T}(\mathbb{R}^3)$, hence $\dim(\mathbf{T}(\mathbb{R}^3)) = 3$. Thus $\mathbf{T}$ is non-singular.

Alternatively,

$$\mathbf{T}(a,b,c) = (0,0,0) \iff a-b=0, b-c=0, a+c=0 \implies a=b=c=0$$

Thus $\mathbf{T}$ is 1-1, therefore it is onto, which shows it is nonsingular. ∎

Question 1(b) *Prove that a square matrix* $\mathbf{A}$ *is non-singular if and only if the constant term in its characteristic polynomial is different from* 0.

Solution. Let

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

Then characteristic polynomial of $\mathbf{A} = \mathbf{Det}(x\mathbf{I} - \mathbf{A})$. $\mathbf{I} = n \times n$ unit matrix. Clearly

$$\mathbf{Det}(x\mathbf{I} - \mathbf{A}) = x^n - \sum_{i=1}^{n} a_{ii} x^{n-1} + \ldots + (-1)^n \mathbf{Det}\mathbf{A}$$

Thus $\mathbf{A}$ is nonsingular iff the constant term in the characteristic polynomial of $\mathbf{A} \neq 0$. ∎

Question 2(a) *Let* $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ *be a linear mapping given by* $T(a,b,c,d,e) = (b-d, d+e, b, 2d+e, b+e)$. *Obtain bases for its null space and range space.*

Solution. Clearly

$$\begin{aligned} T(1,0,0,0,0) &= (0,0,0,0,0) \\ T(0,1,0,0,0) &= (1,0,1,0,1) \\ T(0,0,1,0,0) &= (0,0,0,0,0) \\ T(0,0,0,1,0) &= (-1,1,0,2,0) \\ T(0,0,0,0,1) &= (0,1,0,1,1) \end{aligned}$$

are generators of the range space of T. In fact, if $\mathbf{v_1} = (1,0,1,0,1), \mathbf{v_2} = (-1,1,0,2,0), \mathbf{v_3} = (0,1,0,1,1)$ then $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ generate $T(\mathbb{R}^5)$. We now show that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent. Let $\alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \alpha_3\mathbf{v_3} = 0$. Then $\alpha_1 - \alpha_2 = 0, \alpha_2 + \alpha_3 = 0, \alpha_1 = 0 \Rightarrow \alpha_2 = \alpha_3 = 0$. Thus $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent over $\mathbb{R} \Rightarrow T(\mathbb{R}^5)$ is of dimension 3 with basis $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$.

Thus the null space is of dimension 2, because dim(null space) + dim(range space) = dim(given vector space $= \mathbb{R}^5$) = 5. Since $\mathbf{e_1} = (1,0,0,0,0)$ and $\mathbf{e_3} = (0,0,1,0,0)$ belong to the null space of T, and both are linearly independent over $\mathbb{R}$, $\mathbf{e_1}, \mathbf{e_3}$ is a basis of the null space of T. ∎

Question 2(b) *Let* $\mathbf{A}$ *be a* 3×3 *real symmetric matrix with eigenvalues* $0, 0, 5$. *If the corresponding eigenvectors are* $(2, 0, 1), (2, 1, 1), (1, 0, -2)$ *then find the matrix* $\mathbf{A}$.

Solution. Let $\mathbf{P} = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$, then $\mathbf{P^{-1}AP} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, so $\mathbf{A} = \mathbf{P} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} \mathbf{P^{-1}}$.

A simple calculation shows that $\mathbf{P^{-1}} = \begin{pmatrix} \frac{2}{5} & -1 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & -\frac{2}{5} \end{pmatrix}$, therefore

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & -1 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

Thus $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$ is the required symmetric matrix with $0, 0, 5$ as eigenvalues. ∎

Question 2(c) *Solve the following system of linear equations:*

$$\begin{aligned} x_1 - 2x_2 - 3x_3 + 4x_4 &= -1 \\ -x_1 + 3x_2 + 5x_3 - 5x_4 - 2x_5 &= 0 \\ 2x_1 + x_2 - 2x_3 + 3x_4 - 4x_5 &= 17 \end{aligned}$$

Solution. There are three equations in 5 unknowns, therefore the rank of the coefficient matrix ≤ 3. Since $\begin{vmatrix} 1 & -2 & -3 \\ -1 & 3 & 5 \\ -2 & 1 & -2 \end{vmatrix} = 1(-6-5) + 2(2-10) + (-3)(-1-6) = -6$, the rank of the coefficient matrix is 3. Using Cramer's rule we solve the system

$$\begin{aligned} x_1 - 2x_2 - 3x_3 &= -1 - 4x_4 & (1) \\ -x_1 + 3x_2 + 5x_3 &= 5x_4 + 2x_5 & (2) \\ 2x_1 + x_2 - 2x_3 &= 17 - 3x_4 + 4x_5 & (3) \end{aligned}$$

$$
\begin{aligned}
x_1 &= -\frac{1}{6}\begin{vmatrix} -1-4x_4 & -2 & -3 \\ 5x_4+2x_5 & 3 & 5 \\ 17-3x_4+4x_5 & 1 & -2 \end{vmatrix} \\
&= -\frac{1}{6}[(-1-4x_4)(-11)-3(5x_4+2x_5-51+9x_4-12x_5)+2(-10x_4-4x_5-85+15x_4-20x_5)] \\
&= -\frac{1}{6}[-6+44x_4-42x_4+10x_4+30x_5-48x_5] \\
&= 1-2x_4+3x_5
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= -\frac{1}{6}\begin{vmatrix} 1 & -1-4x_4 & -3 \\ -1 & 5x_4+2x_5 & 5 \\ 2 & 17-3x_4+4x_5 & -2 \end{vmatrix} \\
&= -\frac{1}{6}[-10x_4-4x_5-85+15x_4-20x_5-8-32x_4+51-9x_4+12x_5+30x_4+12x_5] \\
&= -\frac{1}{6}[-42-6x_4] \\
&= 7+x_4
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= -\frac{1}{6}\begin{vmatrix} 1 & -2 & -1-4x_4 \\ -1 & 3 & 5x_4+2x_5 \\ 2 & 1 & 17-3x_4+4x_5 \end{vmatrix} \\
&= -\frac{1}{6}[51-9x_4+12x_5-5x_4-2x_5-34+6x_4-8x_5-20x_4-8x_5+7+28x_4] \\
&= -\frac{1}{6}[24-6x_5] \\
&= -4+x_5
\end{aligned}
$$

The solution space is $(1-2x_4+3x_5, 7+x_4, -4+x_5, x_4, x_5)$, where $x_4, x_5 \in \mathbb{R}$ (arbitrarily). Note that the vector space of solutions is of dimension 2.

Alternate Method.

$$
\begin{aligned}
x_2+2x_3 &= -1+x_4+2x_5 \text{ adding (1) and (2)} && (4) \\
7x_2+8x_3 &= 17+7x_4+8x_5 \text{ adding } 2\times(2) \text{ and (3)} && (5) \\
6x_3 &= -24+6x_5 \text{ using } 7\times(4) \text{ - (5)} && (6) \\
x_2 &= 7+x_4 \text{ from (4) and (6)} && (7) \\
x_1 &= 1-2x_4+3x_5 \text{ using (1), (6), (7)} && (8)
\end{aligned}
$$

The solution space is as shown above. ∎

Question 2(d) *Use Cayley-Hamilton theorem to find the inverse of the following matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$$

Solution. Characteristic polynomial is given by $|x\mathbf{I} - \mathbf{A}| = 0$, where $\mathbf{I}$ is the 3×3 unit matrix.

$$\begin{aligned} \begin{vmatrix} x & -1 & -2 \\ -1 & x-2 & -3 \\ -3 & -1 & x-1 \end{vmatrix} &= 0 \\ x[x^2 - 3x + 2 - 3] + 1[-x + 1 - 9] - 2[1 + 3x - 6] &= 0 \\ x^3 - 3x^2 - 8x + 2 &= 0 \end{aligned}$$

By Cayley-Hamilton theorem, $\mathbf{A}^3 - 3\mathbf{A}^2 - 8\mathbf{A} + 2\mathbf{I} = 0$, or $\mathbf{A}(\mathbf{A}^2 - 3\mathbf{A} - 8\mathbf{I}) = -2\mathbf{I}$. Thus

$$\begin{aligned} \mathbf{A}^{-1} &= -\frac{1}{2}(\mathbf{A}^2 - 3\mathbf{A} - 8\mathbf{I}) \\ &= -\frac{1}{2}\left[\begin{pmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{pmatrix} - \begin{pmatrix} 0 & 3 & 6 \\ 3 & 6 & 9 \\ 9 & 3 & 3 \end{pmatrix} - \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}\right] \\ &= -\frac{1}{2}\begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \end{aligned}$$

Check $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. ∎