UPSC Civil Services Main 2003 - Mathematics Linear Algebra

Sunder Lal

Retired Professor of Mathematics

Panjab University

Chandigarh

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Question 1(a) *Let $\mathcal{S}$ be any non-empty subset of a vector space $\mathcal{V}$ over the field F. Show that the set $\{a_1\mathbf{x_1} + \ldots + a_n\mathbf{x_n} \mid a_1, \ldots, a_n \in F,\ \mathbf{x_1}, \ldots \mathbf{x_n} \in \mathcal{S},\ n \in \mathbb{N}\}$ is the subspace generated by $\mathcal{S}$.*

Solution. Let $\mathcal{W}$ be the subset mentioned above. Let $\mathbf{w_1}, \mathbf{w_2} \in \mathcal{W}$ and $a, b \in F$. Then $\mathbf{w_1} = a_1\mathbf{x_1} + \ldots a_r\mathbf{x_r}$, where $a_1, \ldots, a_r \in F, \mathbf{x_1}, \ldots \mathbf{x_r} \in \mathcal{S}$ and $\mathbf{w_2} = b_1\mathbf{y_1} + \ldots b_s\mathbf{y_s}$ where $b_1, \ldots b_s \in F, \mathbf{y_1}, \ldots, \mathbf{y_s} \in \mathcal{S}$. Now $\alpha\mathbf{w_1} + \beta\mathbf{w_2} = c_1\mathbf{z_1} + \ldots + c_{r+s}\mathbf{z_{r+s}}$, where $c_i = \alpha a_i, 1 \leq i \leq r,\ c_{j+r} = \beta b_j, 1 \leq j \leq s$, and $\mathbf{z_i} = \mathbf{x_i}, 1 \leq i \leq r,\ \mathbf{z_{j+r}} = \mathbf{y_j}, 1 \leq j \leq s$. Clearly $c_j \in F, \mathbf{z_j} \in \mathcal{S}$ for $1 \leq j \leq r+s$, showing that for any $\mathbf{w_1}, \mathbf{w_2} \in \mathcal{W}, \alpha, \beta \in F, \alpha\mathbf{w_1} + \beta\mathbf{w_2} \in \mathcal{W}$, moreover $\mathcal{W} \neq \emptyset$ as $\mathcal{S} \subseteq \mathcal{W}$ and $\mathcal{S} \neq \emptyset$. Thus $\mathcal{W}$ is a subspace of $\mathcal{V}$. ∎

Question 1(b) *If* $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$, *then find the matrix represented by* $2\mathbf{A}^{10} - 10\mathbf{A}^9 + 14\mathbf{A}^8 - 6\mathbf{A}^7 - 3\mathbf{A}^6 + 15\mathbf{A}^5 - 21\mathbf{A}^4 + 9\mathbf{A}^3 + \mathbf{A} - \mathbf{I}$.

Solution. The characteristic equation of $\mathbf{A}$ is

$$|\mathbf{A} - x\mathbf{I}| = \begin{vmatrix} 2-x & 1 & 1 \\ 0 & 1-x & 0 \\ 1 & 1 & 2-x \end{vmatrix} = (2-x)^2(1-x) - (1-x) = 0$$

or $(1-x)(4-4x+x^2) - 1 + x = 3 - 7x + 5x^2 - x^3 = 0$, or $x^3 - 5x^2 + 7x - 3 = 0$. By the

Cayley-Hamilton theorem, we get $\mathbf{A}^3 - 5\mathbf{A}^2 + 7\mathbf{A} - 3\mathbf{I} = \mathbf{0}$. Now

$$\begin{aligned}
& 2\mathbf{A}^{10} - 10\mathbf{A}^9 + 14\mathbf{A}^8 - 6\mathbf{A}^7 - 3\mathbf{A}^6 + 15\mathbf{A}^5 - 21\mathbf{A}^4 + 9\mathbf{A}^3 + \mathbf{A} - \mathbf{I} \\
= \; & 2\mathbf{A}^7[\mathbf{A}^3 - 5\mathbf{A}^2 + 7\mathbf{A} - 3\mathbf{I}] - 3\mathbf{A}^3[\mathbf{A}^3 - 5\mathbf{A}^2 + 7\mathbf{A} - 3\mathbf{I}] + \mathbf{A} - \mathbf{I} \\
= \; & 2\mathbf{A}^7\mathbf{0} - 3\mathbf{A}^3\mathbf{0} + \mathbf{A} - \mathbf{I} \\
= \; & \mathbf{A} - \mathbf{I} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}
\end{aligned}$$

which is the required value. ∎

Question 2(a) *Prove that the eigenvectors corresponding to distinct eigenvalues of a square matrix are linearly independent.*

Solution. Let $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}$ be eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of the square matrix $\mathbf{A}$.

We will show that if any subset of the vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}$ is linearly dependent, then we can find a smaller set that is also linearly dependent — but this leads to a contradiction as the eigenvectors are all non-zero.

Suppose, without loss of generality, that $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_r}$ are linearly dependent. Thus there exist $\alpha_1, \ldots \alpha_r \in \mathbb{R}$, not all zero, such that

$$\alpha_1\mathbf{x_1} + \ldots + \alpha_r\mathbf{x_r} = \mathbf{0} \tag{1}$$

Thus $\mathbf{A}(\alpha_1\mathbf{x_1} + \ldots + \alpha_r\mathbf{x_r}) = \mathbf{0} \Rightarrow \alpha_1\lambda_1\mathbf{x_1} + \ldots + \alpha_r\lambda_r\mathbf{x_r} = \mathbf{0}$. Multiplying (1) by λ_1 and subtracting, we have $\alpha_2(\lambda_2 - \lambda_1)\mathbf{x_2} + \ldots + \alpha_r(\lambda_r - \lambda_1)\mathbf{x_r} = \mathbf{0}$. Now $\alpha_i \neq 0 \Rightarrow \alpha_i(\lambda_i - \lambda_1) \neq 0$, so not all $\alpha_i(\lambda_i - \lambda_1)$ can be zero, so we have a smaller set $\mathbf{x_2}, \ldots, \mathbf{x_r}$ which is also linearly dependent. This leads us to the contradiction mentioned above, hence the vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}$ must be linearly independent. ∎

Question 2(b) *If* $\mathbf{H}$ *is a Hermitian matrix, then show that* $(\mathbf{H} + i\mathbf{I})^{-1}(\mathbf{H} - i\mathbf{I})$ *is a unitary matrix. Also show that every unitary matrix* $\mathbf{A}$ *can be written in this form provided* 1 *is not an eigenvalue of* $\mathbf{A}$.

Solution. See related results of 1989, question 2(b). ∎

Question 2(c) *If* $\mathbf{A} = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ *then find a diagonal matrix* $\mathbf{D}$ *and a matrix* $\mathbf{B}$ *such that* $\mathbf{A} = \mathbf{BDB'}$ *where* $\mathbf{B'}$ *denotes the transpose of* $\mathbf{B}$.

Solution. Let $\mathbb{Q}(x_1, x_2, x_3) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the quadratic form associated with $\mathbf{A}$. Then

$$\begin{array}{rcl} \mathbb{Q}(x_1, x_2, x_3) & = & 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 + 4x_1x_3 - 2x_2x_3 \\ & = & 6[x_1 - \dfrac{1}{3}x_2 + \dfrac{1}{3}x_3]^2 + \dfrac{7}{3}x_2^2 + \dfrac{7}{3}x_3^2 - \dfrac{2}{3}x_2x_3 \\ & = & 6[x_1 - \dfrac{1}{3}x_2 + \dfrac{1}{3}x_3]^2 + \dfrac{7}{3}[x_2 - \dfrac{1}{7}x_3]^2 + \dfrac{16}{7}x_3^2 \end{array}$$

Let $X_1 = x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_3, X_2 = x_2 - \frac{1}{7}x_3, X_3 = x_3$ and $\mathbf{D} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & 0 \\ 0 & 0 & \frac{16}{7} \end{pmatrix}$. Then

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \mathbf{BDB'} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix} \mathbf{D} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

where $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{1}{7} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{B'} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Thus $\mathbf{A} = \mathbf{BDB'}$ where $\mathbf{D} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & 0 \\ 0 & 0 & \frac{16}{7} \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ \frac{1}{3} & -\frac{1}{7} & 1 \end{pmatrix}$ ∎

Question 2(d) *Reduce the quadratic form given below to canonical form and find its rank and signature:*

$$x^2 + 4y^2 + 9z^2 + u^2 - 12yx + 6zx - 4zy - 2xu - 6zu$$

Solution. Let

$$\begin{array}{rcl} \mathbb{Q}(x, y, z, u) & = & x^2 + 4y^2 + 9z^2 + u^2 - 12yx + 6zx - 4zy - 2xu - 6zu \\ & = & (x - 6y + 3z - u)^2 - 32y^2 + 32yz - 12yu \\ & = & (x - 6y + 3z - u)^2 - 32(y^2 - yz + \dfrac{3}{8}yu) \\ & = & (x - 6y + 3z - u)^2 - 32(y - \dfrac{1}{2}z + \dfrac{3}{4}u)^2 + 8z^2 + 18u^2 - 24uz \\ & = & (x - 6y + 3z - u)^2 - 32(y - \dfrac{1}{2}z + \dfrac{3}{4}u)^2 + 8(z - \dfrac{3}{2}u)^2 \end{array}$$

Put

$$
\begin{aligned}
X &= x - 6y + 3z - u \\
Y &= y - \frac{1}{2}z + \frac{3}{4}u \\
Z &= z - \frac{3}{2}u \\
U &= u
\end{aligned}
$$

so that $\mathbb{Q}(x, y, z, u)$ is transformed to $X^2 - 32Y^2 + 8Z^2$. We now put $X^* = X, Y^* = \sqrt{32}Y, Z^* = \sqrt{8}Z, U^* = U$ to get ${X^*}^2 - {Y^*}^2 + Z^{*2}$ as the canonical form of $\mathbb{Q}(x, y, z, u)$.

Rank of $\mathbb{Q}(x, y, z, u) = 3 =$ rank of the associated matrix. Signature of $\mathbb{Q}(x, y, z, u) =$ number of positive squares - number of negative squares $= 2 - 1 = 1$. ∎