UPSC Civil Services Main 2005 - Mathematics Linear Algebra

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Question 1(a) *Find the values of* k *for which the vectors* $(1,1,1,1),(1,3,-2,k),(2,2k-2,-k-2,3k-1)$ *and* $(3,k-2,-3,2k+1)$ *are linearly independent in* $\mathbb{R}^4$.

Solution. The given vectors are linearly independent if the matrix

$$\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 3 & -2 & k\\ 2 & 2k-2 & -k-2 & 3k-1\\ 3 & k+2 & -3 & 2k+1\end{pmatrix}$$

is non-singular.

Now

$$\begin{vmatrix}1 & 1 & 1 & 1\\ 1 & 3 & -2 & k\\ 2 & 2k-2 & -k-2 & 3k-1\\ 3 & k+2 & -3 & 2k+1\end{vmatrix} = \begin{vmatrix}1 & 0 & 0 & 0\\ 1 & 2 & -3 & k-1\\ 2 & 2k-4 & -k-4 & 3k-3\\ 3 & k-1 & -6 & 2k-2\end{vmatrix} = \begin{vmatrix}2 & -3 & k-1\\ 2k-4 & -k-4 & 3k-3\\ k-1 & -6 & 2k-2\end{vmatrix}$$

$$= \begin{vmatrix}2 & -3 & k-1\\ 2k-4 & -k-4 & 3k-3\\ k-5 & 0 & 0\end{vmatrix} = (k-5)[-9k+9+(k-1)(k+4)] \neq 0$$

Clearly $(k-5)[-9k+9+(k-1)(k+4)] = 0 \Leftrightarrow k = 1, 5$. Thus the vectors are linearly independent when $k \neq 1, 5$. ∎

Question 1(b) *Let $\mathcal{V}$ be the vector space of polynomials in x of degree $\leq n$ over $\mathbb{R}$. Prove that the set $\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathcal{V}$. Extend this so that it becomes a basis for the set of all polynomials in x.*

Solution. $\{1, x, x^2, \ldots, x^n\}$ are linearly independent over $\mathbb{R}$ — If $a_0 + a_1x + \ldots + a_nx^n = 0$ where $a_i \in \mathbb{R}, 0 \leq i \leq n$, then we must have $a_i = 0$ for every i because the non-zero polynomial $a_0 + a_1x + \ldots + a_nx^n$ can have at most n roots in $\mathbb{R}$ whereas $a_0 + a_1x + \ldots + a_nx^n = 0$ for every $x \in \mathbb{R}$.

Every polynomial in x of degree $\leq n$ is clearly a linear combination of $1, x, x^2, \ldots, x^n$ with coefficients from $\mathbb{R}$. Thus $\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathcal{V}$.

We shall show that $\mathscr{S} = \{1, x, x^2, \ldots, x^n, x^{n+1}, \ldots\}$ is a basis for the space of all polynomials.

(i) Linear Independence: Let $\{x^{i_1}, \ldots, x^{i_r}\}$ be a finite subset of $\mathscr{S}$. Let $n = \max\{i_1, \ldots, i_r\}$, then $\{x^{i_1}, \ldots, x^{i_r}\}$ being a subset of the linearly independent set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent, which shows the linear independence of $\mathscr{S}$.

(ii) Let f be any polynomial. If degree of f is m, then f is a linear combination of $\{1, x, x^2, \ldots, x^m\}$, which is a subset of $\mathscr{S}$. Thus $\mathscr{S}$ is a basis of $\mathcal{W}$, the space of all polynomials over $\mathbb{R}$. ∎

Question 2(a) *Let $\mathbf{T}$ be a linear transformation on $\mathbb{R}^3$ whose matrix relative to the standard basis of $\mathbb{R}^3$ is $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 2 \\ 3 & 3 & 4 \end{pmatrix}$. Find the matrix of $\mathbf{T}$ relative to the basis $\mathscr{B} = \{(1,1,1), (1,1,0), (0,1,1)\}$.*

Solution. Let the vectors of the given basis be $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. $(\mathbf{T}(\mathbf{v_1}), \mathbf{T}(\mathbf{v_2}), \mathbf{T}(\mathbf{v_3})) = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 2 \\ 3 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 0 \\ 5 & 3 & 4 \\ 10 & 6 & 7 \end{pmatrix}$.

If $(a, b, c) = \alpha\mathbf{v_1} + \beta\mathbf{v_2} + \gamma\mathbf{v_3}$, then $\alpha + \beta = a, \alpha + \beta + \gamma = b, \alpha + \gamma = c$ therefore $\alpha = a - b + c, \beta = b - c, \gamma = b - a$. Consequently

$$\begin{aligned} \mathbf{T}(\mathbf{v_1}) &= 7\mathbf{v_1} - 5\mathbf{v_2} + 3\mathbf{v_3} \\ \mathbf{T}(\mathbf{v_1}) &= 6\mathbf{v_1} - 3\mathbf{v_2} + 0\mathbf{v_3} \\ \mathbf{T}(\mathbf{v_1}) &= 3\mathbf{v_1} - 3\mathbf{v_2} + 4\mathbf{v_3} \end{aligned}$$

This shows that the matrix of $\mathbf{T}$ with respect to given basis $\mathscr{B}$ is $\begin{pmatrix} 7 & 6 & 3 \\ -5 & -3 & -3 \\ 3 & 0 & 4 \end{pmatrix}$ ∎

Question 2(b) *If $\mathbf{S}$ is a skew-Hermitian matrix, then show that $\mathbf{A} = (\mathbf{I} + \mathbf{S})(\mathbf{I} - \mathbf{S})^{-1}$ is a unitary matrix. Show that a unitary matrix $\mathbf{A}$ can be expressed in the above form provided -1 is not an eigenvalue of $\mathbf{A}$.*

Solution. See related results of question 2(a) year 1989. ∎

Question 2(c) *Reduce the quadratic form*

$$6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 4x_2x_3 + 4x_1x_3$$

to a sum of squares. Also find the corresponding linear transformation, index and signature.

Solution.

$$\begin{aligned}
\mathcal{Q}(x_1, x_2, x_3) &= 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 4x_2x_3 + 4x_1x_3 \\
&= 6[x_1^2 - \frac{2}{3}x_1x_2 + \frac{2}{3}x_1x_3 + \frac{1}{9}x_2^2 + \frac{1}{9}x_3^2 - \frac{2}{9}x_2x_3] \\
&\quad + \frac{7}{3}x_2^2 + \frac{7}{3}x_3^2 - \frac{8}{3}x_2x_3 \\
&= 6[x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_3]^2 + \frac{7}{3}[x_2 - \frac{4}{7}x_3]^2 + \frac{33}{21}x_3^2
\end{aligned}$$

Put $X_1 = x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_3, X_2 = x_2 - \frac{4}{7}x_3, X_3 = x_3$, so that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{7} \\ 0 & 1 & \frac{4}{7} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \tag{1}$$

and $\mathcal{Q}(x_1, x_2, x_3)$ is transformed to $6X_1^2 + \frac{7}{3}X_2^2 + \frac{33}{21}X_3^2$. Let

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}}Z_1 \\ \sqrt{\frac{3}{7}}Z_2 \\ \sqrt{\frac{7}{11}}Z_3 \end{pmatrix}$$

then $\mathcal{Q}(x_1, x_2, x_3)$ is transformed to $Z_1^2 + Z_2^2 + Z_3^2$, which is its canonical form. Thus $\mathcal{Q}(x_1, x_2, x_3)$ is positive definite. The Index of $\mathcal{Q}(x_1, x_2, x_3) =$ Number of positive squares in its canonical form $= 3$. The signature of $\mathcal{Q}(x_1, x_2, x_3) =$ Number of positive squares - the number of negative squares in its canonical form $= 3$.

The required linear transformation which transforms $\mathcal{Q}(x_1, x_2, x_3)$ to sums of squares is given by (1), and the linear transformation which transforms it to its canonical form is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{7} \\ 0 & 1 & \frac{4}{7} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{7}} & 0 \\ 0 & 0 & \sqrt{\frac{7}{11}} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}$$

∎