

UPSC Civil Services Main 2006 - Mathematics

Linear Algebra

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Question 1(a) Let \mathcal{V} be a vector space of all 2×2 matrices over the field F . Prove that \mathcal{V} has dimension 4 by exhibiting a basis for \mathcal{V} .

Solution. Let $\mathbf{M}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{M}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{M}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\mathbf{M}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We will show that $\{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4\}$ is a basis of \mathcal{V} over F .

$\{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4\}$ generate \mathcal{V} . Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{V}$. Then $\mathbf{A} = a\mathbf{M}_1 + b\mathbf{M}_2 + c\mathbf{M}_3 + d\mathbf{M}_4$, where $a, b, c, d \in F$. Thus $\{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4\}$ is a set of generators for \mathcal{V} over F .

$\{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4\}$ are linearly independent over F . If $a\mathbf{M}_1 + b\mathbf{M}_2 + c\mathbf{M}_3 + d\mathbf{M}_4 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{0}$ for $a, b, c, d \in F$, then clearly $a = b = c = d = 0$, showing that $\{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4\}$ are linearly independent over F .

Hence $\{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4\}$ is a basis of \mathcal{V} over F and $\dim \mathcal{V} = 4$. ■

Question 1(b) State the Cayley-Hamilton theorem and using it find the inverse of $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

Solution. Let \mathbf{A} be an $n \times n$ matrix and let \mathbf{I}_n be the $n \times n$ identity matrix. Then the n -degree polynomial $|\mathbf{xI}_n - \mathbf{A}|$ is called the characteristic polynomial of \mathbf{A} . The Cayley-Hamilton theorem states that every matrix is a root of its characteristic polynomial:

$$\begin{aligned} \text{if } & |\mathbf{xI}_n - \mathbf{A}| = x^n + a_1x^{n-1} + \dots + a_n \\ \text{then } & \mathbf{A}^n + a_1\mathbf{A}^{n-1} + \dots + a_n\mathbf{I}_n = \mathbf{0} \end{aligned}$$

$|\mathbf{xI}_n - \mathbf{A}| = 0$ is called the characteristic equation of \mathbf{A} .

Let $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$. The characteristic equation of \mathbf{A} is $0 = \begin{vmatrix} x-1 & -3 \\ -2 & x-4 \end{vmatrix} = (x-1)(x-4) - 6 = x^2 - 5x - 2$.

By the Cayley-Hamilton Theorem, $\mathbf{A}^2 - 5\mathbf{A} - 2\mathbf{I}_2 = \mathbf{0} \Rightarrow \mathbf{A}(\mathbf{A} - 5\mathbf{I}_2) = (\mathbf{A} - 5\mathbf{I}_2)\mathbf{A} = 2\mathbf{I}_2$.

Thus \mathbf{A} is invertible and $\mathbf{A}^{-1} = \frac{1}{2}(\mathbf{A} - 5\mathbf{I}_2)$, so $\mathbf{A}^{-1} = \frac{1}{2} \left[\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \right] = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$ ■

Question 2(a) If $\mathbf{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by $\mathbf{T}(x, y) = (2x - 3y, x + y)$, compute the matrix of \mathbf{T} with respect to the basis $\mathcal{B} = \{(1, 2), (2, 3)\}$.

Solution. It is obvious that $\mathbf{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation. Clearly

$$\begin{aligned}\mathbf{T}(\mathbf{v}_1) &= \mathbf{T}(1, 2) = (-4, 3) \\ \mathbf{T}(\mathbf{v}_2) &= \mathbf{T}(2, 3) = (-5, 5)\end{aligned}$$

Let $(a, b) = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2$, where $a, b, \alpha, \beta \in \mathbb{R}$, then $\alpha + 2\beta = a, 2\alpha + 3\beta = b \Rightarrow \alpha = 2b - 3a, \beta = 2a - b$. Thus $\mathbf{T}(\mathbf{v}_1) = 18\mathbf{v}_1 - 11\mathbf{v}_2, \mathbf{T}(\mathbf{v}_2) = 25\mathbf{v}_1 - 15\mathbf{v}_2$, so $(\mathbf{v}_1, \mathbf{v}_2)\mathbf{T} = (\mathbf{T}(\mathbf{v}_1), \mathbf{T}(\mathbf{v}_2)) = (\mathbf{v}_1, \mathbf{v}_2) \begin{pmatrix} 18 & 25 \\ -11 & -15 \end{pmatrix}$. Thus the matrix of \mathbf{T} with respect to the basis \mathcal{B} is $\begin{pmatrix} 18 & 25 \\ -11 & -15 \end{pmatrix}$ ■

Question 2(b) Using elementary row operations, find the rank of $\mathbf{A} = \begin{pmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$

Solution. Operations $\mathbf{R}_1 - 2\mathbf{R}_3, \mathbf{R}_2 - \mathbf{R}_4$ give

$$\mathbf{A} \sim \begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 1 & 0 & 0 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Operation $\mathbf{R}_3 - \mathbf{R}_1$ gives

$$\mathbf{A} \sim \begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & -9 & -5 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Operations $\mathbf{R}_3 + 4\mathbf{R}_2, \mathbf{R}_4 - \mathbf{R}_2 \Rightarrow$

$$\mathbf{A} \sim \begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -9 & -5 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

$\mathbf{R}_4 + \frac{2}{9}\mathbf{R}_3 \Rightarrow$

$$\mathbf{A} \sim \begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -9 & -5 \\ 0 & 0 & 0 & -\frac{1}{9} \end{pmatrix}$$

Clearly $|\mathbf{A}| = 1 \Rightarrow \text{rank } \mathbf{A} = 4$. ■

Question 2(c) Investigate for what values of λ and μ the equations

$$\begin{aligned}x + y + z &= 6 \\x + 2y + 3z &= 10 \\x + 2y + \lambda z &= \mu\end{aligned}$$

have (1) no solution (2) a unique solution (3) infinitely many solutions.

Solution. (2) The equations will have a unique solution for all values of μ if the coefficient matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{pmatrix}$ is non-singular. i.e. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & \lambda - 1 \end{vmatrix} = \lambda - 1 - 2 \neq 0$ i.e. $\lambda \neq 3$.

Thus for $\lambda \neq 3$ and for all μ we have a unique solution which can be obtained by Cramer's rule or otherwise.

(1) If $\lambda = 3, \mu \neq 10$ then the system is inconsistent and we have no solution.

(3) If $\lambda = 3, \mu = 10$, the system will have infinitely many solutions obtained by solving $x + y = 6 - z, x + 2y = 10 - 3z \Rightarrow x = 2 + z, y = 4 - 2z, z$ is any real number. ■

Question 2(d) Find the quadratic form $q(x, y)$ corresponding to the symmetric matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -3 \\ -3 & 8 \end{pmatrix}$$

Is this quadratic form positive definite? Justify your answer.

Solution. The quadratic form is

$$\begin{aligned}q(x, y) &= (x \ y) \begin{pmatrix} 5 & -3 \\ -3 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\&= 5x^2 - 6xy + 8y^2 \\&= 5\left[x^2 - \frac{6}{5}xy + \frac{8}{5}y^2\right] \\&= 5\left[\left(x - \frac{3}{5}y\right)^2 + \frac{31}{25}y^2\right]\end{aligned}$$

Clearly $q(x, y) > 0$ for all $(x, y) \neq (0, 0), (x, y) \in \mathbb{R}^2$. Thus $q(x, y)$ is positive definite. In fact, $q(x, y) = 0 \Rightarrow x - \frac{3}{5}y = 0, y = 0 \Rightarrow x = y = 0$. ■